Goldberg’s constant and its relatives

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Let \( f \) be a holomorphic function in the unit disc having exactly one simple zero \( z_0 \) and two 1-points (counting multiplicity) \( z_1, z_2 \). Let \( \rho \) be the radius of the smallest hyperbolic disc containing \( z_0, z_1, z_2 \). Goldberg [4] proved that \( \rho \geq \mu > 0 \), where \( \mu \) is an absolute constant. What is the best (largest value) of \( \mu \) for which this is true? This optimal value is called the Goldberg constant \( A_2 \).

There is a conjecture that the extremal function must have one double 1-point. Assuming this, the extremal function was found in [1].

There are several similar problems.

1. Let \( f \) be a function holomorphic in the unit disc, \( f(0) = 0 \), \( f'(0) \neq 0 \), and no other zeros, and such that the equation \( f(z) = 1 \) has two solutions, \( z_1, z_2 \) counting multiplicity.
   a) What is the minimal value of \( \max\{|z_1|, |z_2|\} \)?
   b) What is the maximal value of \( |f'(0)| \)?

2. Same questions with the additional assumption that \( f \) is real.

3. Let \( f \) be a real holomorphic function in the unit disc, having one simple zero at 0 and two simple 1-points at \( \pm ia \). What is the minimal value of \( a \)? This minimal value is called the Belgian Chocolate Constant because the prize of 1kg of fine Belgian chocolate is offered for it [2, p. 149f].

4. All these problems can be also asked for rational functions of given degree. I conjecture that the extremal function for Problems 1,2 and for the Goldberg constant is a Belyi function with 1-point of multiplicity 2.

All these problems are relevant for control theory [2, 3].
References


[3] A. Eremenko, Simultaneous stabilization, avoidance and Goldberg’s constants,
http://www.math.purdue.edu/ eremenko/dvi/control2.pdf