A remarkable identity

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Let h and p be two polynomials. When $y = p(x)e^{h(x)}$ satisfies a second order differential equation

$$y'' + Py = 0, \tag{1}$$

where P is a polynomial? Substitution gives

$$\frac{p''}{p} + 2\frac{p'}{p}h' + h'' + h'^2 + P = 0. \tag{2}$$

Such P exists if and only if $p'' + 2p'h'$ is divisible by p.

Another criterion is obtained if we consider the second solution y_1 of (1) which is linearly independent of y. This second solution can be found from the condition

$$yy_1' - y'y_1 = 1. \tag{3}$$

Solving (3) with respect to y_1 we obtain

$$y_1 = pe^h \int p^{-2}e^{-2h}. \tag{4}$$

As all solutions of (1) must be entire functions, we conclude that all residues of $p^{-2}e^{-2h}$ must vanish. This condition is *necessary and sufficient* for $y = pe^h$ to satisfy equation (1) with some P. Indeed, if y_1 defined by (4) is entire, then y and y_1 is a pair of entire functions whose Wronski deterinant is 1, so this pair must satisfy a differential equation (1) with entire P and asymptotics at infinity show that P must be a polynomial.

Now suppose that h is an odd polynomial of degree 3, which we write in the form

$$h(x) = x^3/3 + bx. \tag{5}$$

Suppose that all residues of $p^{-2}e^{-2h}$ vanish. Then the integral $\int p^{-2}e^{-2h}$ is a meromorphic function in the plane. Surprisingly, the integral of some linear combination

$$\int \left(p^2(-x)e^{-2h(x)} - cp^{-2}(x)e^{-2h(x)}\right)$$

is not only meromorphic but is an *elementary function*! Here c is a constant depending on p.

Conjecture. *Let h be given by (5). Let p be a polynomial. All residues of $p^{-2}e^{-2h}$ vanish if and only if there exist a constant c and a polynomial q such that*

$$\left(p^2(-x) - \frac{c}{p^2(x)}\right)e^{-2h(x)} = \frac{d}{dx}\left(\frac{q(x)}{p(x)}e^{-2h(x)}\right).$$

In other words:

$$p^2(x)p^2(-x) - c = q'(x)p(x) - q(x)p'(x) - 2q(x)p(x)h'(x).$$

It is known [1] that for given h of the form (5) there exist polynomials p of any given degree such that all residues of $p^{-2}e^{-2h}$ vanish. These polynomials p have simple roots. We verified the Conjecture for $1 \leq n \leq 4$ where $n = \deg p$ by symbolic computation using Maple. We don't know whether there is any analog of this conjecture for other polynomials h.

Substituting $p(x) = x^n + ax^{n-1} + \ldots$ into (2) and using (5), we conclude that

$$P(x) = -h'^2 - h'' - 2nx + 2a = -x^4 - 2x^2b - 2(n+1)x - b^2 + 2a. \quad (6)$$

Substituting $y = p_n e^h$ with h as in (5) to (1) with this P we obtain a polynomial relation between b and $\lambda := b^2 - 2a$,

$$Q_n(b, \lambda) = 0. \quad (7)$$

We have $\deg_\lambda Q_n = n+1$, [1]. For every b and every λ satisfying this equation, the differential equation (1), with P as in (6), has a solution $y = p_n e^h$ where $\deg p_n = n$.

Functions $y_n = p_n e^h$ are eigenfunctions of the operator

$$y'' - (x^4 + 2bx^2 + 2(n+1)x)y \quad (8)$$

with eigenvalue λ. For each non-negative integer n, and generic b, the operator (8) has $n+1$ eigenfunctions of the form $p_n e^h$ with eigenvalues λ which are solutions of (7).

We assume that Q_n is monic as a polynomial of λ, and p_n is a monic polynomial of x. Constant c in the Conjecture, turns out to be

$$c(b,\lambda) = (-1)^n 2^{-2n} \frac{\partial}{\partial \lambda} Q_n.$$

This is also confirmed only by symbolic computation for small n.

References

[1] C. Bender and S. Boettcher, Quasi-exactly solvable quartic potential, arXiv:physics/9801007,