An estimate of density of zeros

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Let f be an entire function of exponential type with indicator diagram $[-i\pi, i\pi]$. Let $n(r)$ be the counting function of zeros of f.

Give a good estimate from above for the density

$$D = \limsup_{r \to \infty} \frac{n(r)}{r}.$$

The "trivial" estimate using Jensen's formula is $D \leq 2e$. Taking $f(z) = \sin \pi z$ we obtain $D = 2$. An example with $D > 2$ is contained in [1].

Here is an equivalent formulation. Consider a region G of the form $G = \{x + iy : y > \phi(x)\}$, where $\phi \leq 0$, $\phi(0) = 0$, and let F be a conformal map of the upper half-plane onto G such that $f(0) = 0$ and $f(z) \sim z$, $z \to \infty$. Give a good estimate from above for $\text{Re} f(1)$.

References

[1] A. Eremenko and D. Novikov, Oscillation of Fourier integrals with a spectral gap, J. de Math. Pures et Appl., 8, 3 (2004), 313-365.