

Collision of a jet with a plane

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Consider a jet symmetric with respect to the x -axis, the ideal fluid moving left to right with velocity v_∞ at $-\infty$. Let $2r$ be the width of the jet. Suppose that the jet collides with a plane intersecting the x -axis at $x = 0$ and inclined to the x -axis at the angle $\alpha \in (0, \pi/2]$. We want to find the stream lines and the pressure by the jet on the plane. The outside pressure is p_0 , a given constant, and we neglect all other external forces.

The region occupied by the jet will be denoted by D . The boundary ∂D consists of the upper and lower curves L^+ and L^- and the line L whose equation is $y = x \tan \alpha$. The stream ramifies at some point O on this line: one part slides up another down and back along the line L .

Let F be the complex potential of the flow. It is defined up to an additive constant; we choose this constant so that $F(O) = 0$. Then F maps conformally the region D in the z -plane onto the horizontal strip with a horizontal cut in the w -plane:

$$D' = \{w : -q' < |\Im w| < q\} \setminus \{w : w \geq 0\},$$

with some $q, q' > 0$. Let us find the constants q, q' . We have $F(z) \sim v_\infty z$ as $z \rightarrow -\infty$ in D , so

$$q + q' = 2rv_\infty. \tag{1}$$

Let a_1 and a_2 be the limit widths of the streams “up” and “down” along L , and v_1, v_2 their limit velocities at infinity. We obtain as before $q = a_1 v_1$, $q' = a_2 v_2$. Now

$$q - q' = a_1 v_2 - a_2 v_2 = 2rv_\infty \cos \alpha,$$

because the moment along the line L must be preserved: the jets slide along L without friction. From these equations we find

$$q = rv_\infty(1 + \cos \alpha), \quad q' = rv_\infty(1 - \cos \alpha).$$

Recall that $f = F'$ is the complex velocity. The pressure outside D is constant, so by Bernoulli's law the speed must be constant, so we obtain

$$|F'(z)| = v_\infty, \quad z \in L^+ \cup L^-. \quad (2)$$

On the line L we have

$$\arg F'(z) = \begin{cases} -\alpha & \text{on the upper half} \\ \pi - \alpha & \text{on the lower half.} \end{cases} \quad (3)$$

Let us consider the inverse function $G : D' \rightarrow D$, and $g = \log G'$. Then we have from (2) and (3):

$$\Re g(w) = -\log v_\infty, \quad \Im w \in \{q, -q'\}, \quad (4)$$

and

$$\begin{aligned} \Im g(w) &= \alpha, & w &= u + i0, & u &> 0, \\ \Im g(w) &= \alpha - \pi, & w &= u - i0, & u &> 0. \end{aligned} \quad (5)$$

This is called a *mixed boundary value problem*, the real part of the function is prescribed on one part of the boundary and the imaginary part on the complementary part.

A solution of the mixed boundary value problem with piecewise constant boundary values can be obtained by the M.V. Keldysh¹ and L.I. Sedov method as follows: just take for g the conformal map of D' onto a region bounded by the appropriate vertical and horizontal lines!

A little thinking (in which order the arcs are traced) shows that this region must be the half-strip

$$S = \{\zeta : \alpha - \pi < \Im \zeta < \alpha, \Re \zeta > -\log v_0\}.$$

This is a nice conformal map exercise. The boundary correspondence is as follows: The two “right infinities” in D' go to the 90° corners of S , and the point $0 \in \partial D'$ goes to the infinity in ∂S .

Such map always exists by the Riemann mapping theorem, and I leave it to you to find it explicitly and to compute the pressure of the jet on the line L .

The question of uniqueness of solution, of course has not been addressed at all in this text...

¹Besides his fine achievements in complex function theory, Keldysh also initiated the Soviet space program and was its main leader.