Derivation of Kepler's laws from the inverse square law (following V. I. Arnold)

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We use complex numbers and represent the motion as $z(t)$. We first derive the equal area law for an object moving under any central force:

$$z'' = zf(|z|)$$

Let $z = re^{i\phi}$. Then

$$z'' = r''e^{i\phi} + 2ir'\phi'e^{i\phi} + ir\phi''e^{i\phi} - r\phi'^2 e^{i\phi} = re^{i\phi}f(r).$$

Dividing by $e^{i\phi}$ and taking the imaginary part we obtain

$$2r'\phi' + r\phi'' = 0.$$

But this is the derivative of $r^2\phi'$ times r. So the area changes at a constant rate.

Now we derive the first law. Consider an ellipse centered at zero.

Theorem. Squaring sends it to an ellipse with a focus at zero.

Proof. Parametrize the ellipse by Joukowski function: $w = z + 1/z, |z| = r, r > 1$. The foci are ± 2. Then $w^2 = z^2 + 1/z^2 + 2$, an ellipse with foci ± 2 shifted by 2.

Corollary. Each ellipse with a focus at 0 is a square of the unique ellipse with center at 0.

Theorem. The differential equation $z'' = -z$ after the transformation $w = z^2$ and a time change becomes

$$w'' = -cw/|w|^3. \tag{1}$$

Proof. Choose new time τ so that $d\tau/dt = |w|^2/|z|^2 = |z|^2$. Then $d/d\tau = |z|^{-2}d/dt$. Now we compute:

$$
\begin{aligned}
d^2w/d\tau^2 &= |z|^{-2}\frac{d}{dt}|z|^{-2}\frac{d}{dt}z^2 = 2|z|^{-2}\frac{d}{dt}\frac{1}{\overline{z}}\frac{dz}{dt} \\
&= -2|z|^{-2}\left(\frac{1}{\overline{z}^2}\frac{d\overline{z}}{dt}\frac{dz}{dt} + \frac{z}{\overline{z}}\right) = -\frac{2}{z\overline{z}^3}(|z'|^2 + |z|^2).
\end{aligned}
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Differentiating the last expression in parentheses and using $z'' = -z$ we obtain that it is constant. So the right hand side is a positive constant times $-1/(z\overline{z}^3) = -w/\overline{w}^3$.

The hint for the time change is that the equal area law must be observed.

Third Kepler Law. Consider the scaling in space and time: $w(t) = au(bt)$. Then u will obey (1) if and only if $a^3b^2 = 1$.

References

[1] V. I. Arnold, Huygens and Barrow, Newton and Hook, Moscow, Nauka, 1989.