

# Derivation of Kepler's laws from the inverse square law (following V. I. Arnold)

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We first derive the equal area law for an object moving under any central force:

$$z'' = zf(|z|)$$

Let  $z = re^{i\phi}$ . Then

$$z'' = r''e^{i\phi} + 2ir'\phi'e^{i\phi} + ir\phi''e^{i\phi} - r\phi'^2e^{i\phi} = re^{i\phi}f(r).$$

Dividing by  $e^{i\phi}$  and taking the imaginary part we obtain

$$2r'\phi' + r\phi'' = 0.$$

But this is the derivative of  $r^2\phi'$ . So the area is preserved.

Now we derive the first law. Consider an ellipse centered at zero.

Theorem. Squaring sends it to an ellipse with a focus at zero.

Proof. Parametrize the ellipse by Joukowski function:  $w = z + 1/z$ ,  $|z| = r$ ,  $r > 1$ . The foci are  $\pm 2$ . Then  $w^2 = z^2 + 1/z^2 + 2$ , an ellipse with foci  $\pm 2$  shifted by 2.

Corollary. Each ellipse with a focus at 0 is a square of the unique ellipse with center at 0.

Theorem. The differential equation  $z'' = -z$  after the transformation  $w = z^2$  and a time change becomes

$$w'' = -cw/|w|^3. \tag{1}$$

*Proof.* Choose new time  $\tau$  so that  $d\tau/dt = |w|^2/|z|^2 = |z|^2$ . Then  $d/ds\tau = |z|^{-2}d/dt$ . Now we compute:

$$\begin{aligned} d^2w/d\tau^2 &= |z|^{-2} \frac{d}{dt} |z|^{-2} \frac{d}{dt} z^2 = 2|z|^{-2} \frac{d}{dt} \frac{1}{\bar{z}} \frac{dz}{dt} \\ &= -2|z|^{-2} \left( \frac{1}{\bar{z}^2} \frac{d\bar{z}}{dt} \frac{dz}{dt} + \frac{z}{\bar{z}} \right) = -2z^{-1} \frac{1}{\bar{z}^3} (|\dot{z}|^2 + |z|^2). \end{aligned}$$

Differentiating the last expression in parantheses and using  $z'' = -z$  we obtain that it is constant. So the right hand side is a positive constant times  $-1/(z\bar{z}^3) = -w\bar{w}^{-3}$ .

The justification of time change is that the equal area law is observed.

Third Kepler Law. Consider the scaling in space and time:  $w(t) = au(bt)$ . Then  $u$  will obey (1) if and only if  $a^3b^2 = 1$ .

## References

- [1] V. I. Arnold, Huygens and Barrow, Newton and Hook, Moscow, Nauka, 1989.