Tangencies between holomorphic maps and holomorphic laminations

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Abstract

We prove that the set of leaves of a holomorphic lamination of codimension one that are tangent to a germ of a holomorphic map is discrete.

Let F be a holomorphic lamination of codimension one in an open set V in a complex Banach space B. In this paper, this means that $V = W \times \mathbf{C}$, where W is a neighborhood of the origin in some Banach space, and the leaves L_λ of the lamination are disjoint graphs of holomorphic functions $\mathbf{w} \mapsto f(\lambda, \mathbf{w})$, $W \to \mathbf{C}$. For holomorphic functions in a Banach space we refer to [5]. Here λ is a parameter and we assume that the dependence of f on λ is continuous. A natural choice of this parameter is such that $\lambda = f(\lambda, 0)$, in which case the continuity with respect to λ follows from the so-called λ-lemma of Mane-Sullivan-Sad and Lyubich, see, for example [5]. With this choice of the parameter, our definition of a lamination coincides with that of a holomorphic motion of $\mathbf{C}$ parametrized by W.

Let $\gamma : U \to V$ be a holomorphic map, $U \subset \mathbf{C}^n$. We say that γ is tangent to the lamination at a point $\mathbf{z}_0 \in U$ if the image of the derivative $\gamma'(\mathbf{z}_0)$ is contained in the tangent space $T_L(\gamma(\mathbf{z}_0))$, where L is the leaf passing through $\gamma(\mathbf{z}_0)$. A leaf for which this holds is called a tangent leaf to γ.

Theorem. *Let K be a compact subset of U. Then the set of leaves tangent to γ at the points of K is finite.*

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For the case of holomorphic curves ($n = 1$) this result is contained in [1, Lemma 9.1] where it is credited to Douady. Artur Avila, in a conversation with the authors, proposed to extend this result to arbitrary holomorphic maps. According to Avila, this generalization has several applications to holomorphic dynamics.

Proof. We assume without loss of generality that

$$f(0, \mathbf{w}) \equiv 0,$$

and that L_0 is tangent to γ at $\mathbf{z}_0 = 0$.

We have to show that other tangent leaves cannot accumulate to L_0. Suppose the contrary, that is suppose that there is a sequence $\lambda_k \to 0$ such that L_{λ_k} are tangent to γ, and let L_{λ_k} be the graphs of the functions $f_k(\mathbf{w}) = f(\lambda_k, \mathbf{w})$. We may assume that tangency points $\mathbf{z}_k \to 0$.

We make several preliminary reductions.

1. Let

$$\gamma(\mathbf{z}) = (\phi(\mathbf{z}), \psi(\mathbf{z})) \in W \times \mathbf{C}.$$

Consider the new lamination in $U \times \mathbf{C}$ whose leaves are the graphs of $f^*(\lambda, \mathbf{z}) = f(\lambda, \phi(\mathbf{z}))$ and the new map $\gamma^*(\mathbf{z}) = (\mathbf{z}, \psi(\mathbf{z}))$. Then γ^* is tangent to a leaf L^* if and only if γ is tangent to L. This reduces our problem to the case that W is an open set in $\mathbf{C}^n$ and the map γ is a graph of a function ψ of the same variable as the functions f_k. From now on we assume that $U = W$ and $\gamma(\mathbf{w}) = (\mathbf{w}, \psi(\mathbf{w}))$.

2. Now we reduce the problem to the case that ψ is a monomial. For this we use the desingularization theorem of Hironaka [4, 2, 3, 6].

Let X be a complex analytic manifold, and ψ an analytic function on X. Then there exists a complex analytic manifold M and a proper surjective map $\pi : M \to X$ such that the restriction of π onto the complement of the π-preimage of the set $\{\psi = 0,\ \psi' = 0\}$ is injective and for each point $\mathbf{z}_0 \in \pi^{-1}(\{\psi = 0\})$ there is a local coordinate system with the origin at $\mathbf{z}_0$ such that $\psi \circ \pi$ is a monomial $z_1^{m_1} \ldots z_n^{m_n}$.

Let $Y = W \times \mathbf{C}$, and let $S \subset Y$ be the set of points $(\mathbf{w}, t)$ with $t \neq 0$ such that the graph of $t = \psi(\mathbf{w})$ is tangent to the lamination. In our proof by contradiction, we assume that the origin belongs to the closure of S. Let $N = M \times \mathbf{C}$, and let $\rho : N \to Y$ be the map defined by $\rho(\mathbf{z}, t) = (\pi(\mathbf{z}), t)$. Then $\rho^{-1}(F)$ is the lamination whose leaves are the components of the ρ-preimages of the leaves of F, and the set $T = \rho^{-1}(S)$ has a limit point $(\mathbf{z}_0, 0)$

with $\pi(\mathbf{z}_0) = 0$ since π is proper. Also the set T is exactly the set of those points in N where the graph $t = \psi \circ \pi(\mathbf{z})$ is tangent to the lamination $\rho^{-1}(F)$ since ρ is injective in a neighborhood of each point of T. (Any point $(\mathbf{z}, t)$ where ρ is not injective satisfies $\psi(\pi(\mathbf{z})) = 0$ while at every point of T we have $\psi(\pi(\mathbf{z})) \neq 0$.) This reduces our problem to the case that ψ is a monomial.

3. We may assume now that $W = \{\mathbf{z} : |\mathbf{z}| < 2\}$. So we are in the following situation.

$$\psi(z) = z_1^{m_1} \dots z_n^{m_n},$$

and $\{f_k\}$ is a family of holomorphic functions on W with disjoint graphs, $f_k(\mathbf{z}) \neq 0$ for $\mathbf{z} \in W$, and $f_k \to 0$ as $k \to \infty$ uniformly on W. Moreover, for some sequence $\mathbf{z}_k \to 0$ we have

$$f_k(\mathbf{z}_k) = \prod_{j=1}^{n} z_{j,k}^{m_j}, \tag{1}$$

$$\text{grad } f_k(\mathbf{z}_k) = (m_1 z_{1,k}^{m_1 - 1} z_{2,k}^{m_2} \dots,\ m_2 z_1^{m_1} z_{2,k}^{m_2 - 1} \dots,\ \dots) \tag{2}$$

assuming zero values for the components with $m_j = 0$. We may assume that $|f_k| \leq 1$ in W. Setting $f_k = \exp g_k$ we obtain that $\Re g_k \leq 0$. Now we put

$$h_k(\mathbf{z}) = g_k(\mathbf{z} + \mathbf{z}_k) - g_k(\mathbf{z}_k).$$

Then the h_k are defined in the unit ball and satisfy

$$\Re h_k(\mathbf{z}) \leq \sum_{j : m_j > 0} m_j \log |z_{j,k}|^{-1}.$$

From this we conclude that

$$\left| \frac{\partial h_k}{\partial z_j}(0) \right| \leq 2 \sum_{j : m_j > 0} m_j \log |z_{j,k}|^{-1}. \tag{3}$$

This follows from the

Lemma. *Let h be an analytic function in the unit disc, $h(0) = 0$, and $\Re h \leq A$, where $A > 0$. Then $|h'(0)| \leq 2A$.*

This is an immediate consequence of the Schwarz Lemma.

On the other hand, (1) and (2) imply that, for $m_j > 0$,

$$\left| \frac{\partial h_k}{\partial z_j}(0) \right| = \frac{m_j}{|z_{j,k}|}. \tag{4}$$

Assume without loss of generality that $m_1 > 0$ and

$$|z_{1,k}| = \min_{j:m_j>0} |z_{j,k}|.$$

Then the RHS of (3) is at most

$$\text{const} \log |z_{1,k}|^{-1},$$

while the RHS of (4) is

$$\frac{m_1}{|z_{1,k}|}.$$

As $|z_{1,k}| \to 0$, we obtain a contradiction which proves our theorem.

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