On the number of solutions of a transcendental equation arising in the theory of gravitational lensing

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Abstract

The equation in the title describes the number of bright images of a point source under lensing by an elliptic object with isothermal density. We prove that this equation has at most 6 solutions. Any number of solutions from 1 to 6 can actually occur.

1. Introduction

We study the number of solutions of the equation

$$z - \frac{k}{\sin \overline{z}} = w, \tag{1}$$

in the region $|\operatorname{Re} z| < \pi/2$ in the complex plane, where $w \in \mathbf{C}$ and $k > 0$ are parameters.

This equation occurs as a model of gravitational lensing of a point source w by an elliptic object whose density equals c/r on the homothetic ellipses rE where E is a fixed ellipse, cf. [5, 7, 8]. As in [5, 8] we shall assume that the density is zero outside of E and thus is equal to c/r on rE only for $0 < r \leq 1$. We note, however, that in the astronomy literature (cf., e. g., [7]) it is usually assumed that this formula for the density holds for $0 < r < \infty$; see [8] for a

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discussion of the different models. We also refer to the above papers for the derivation of equation (1).

The solutions of (1) that lie outside E correspond to the so-called bright images of the source. Khavinson and Lundberg [8] proved that the number of solutions of (1) in $|\operatorname{Re} z| < \pi/2$ is finite and does not exceed 8. Up to 4 bright images by a single lensing galaxy have been observed by astronomers [8, 1].

In this paper we prove the following result.

Theorem. *The number n of solutions of equation* (1) *in the region* $|\operatorname{Re} z| < \pi/2$ *satisfies* $n \leq 6$.

Any number between 1 and 6 can actually occur. In fact, for each value of the parameter k we will describe a partition of the w-plane into regions where the number of solutions is constant. This also yields that equation (1) has at least one solution for every $k > 0$ and $w \in \mathbf{C}$, but we do not include a formal proof of this statement.

The upper estimate in [8] is based on a miraculous trick: an application of Fatou's theorem from holomorphic dynamics. This method originated in [10], and it was applied by Khavinson and Neumann [9] to obtain the estimate $5d - 5$ for the number of solutions of the equation

$$z - R(\overline{z}) = 0,$$

where R is a rational function of degree d. This equation with

$$R(z) = \sum_{j=1}^{d} \frac{a_j}{z - z_j}, \quad a_j > 0,$$

describes gravitational lensing by d coplanar point masses. The estimate $5d - 5$ for the number of solutions is exact, even for this special form of R; this fact was established by S. Rhie [12, 13], who also conjectured the correct estimate $5d - 5$.

Our method does not use holomorphic dynamics. Instead it is based on elementary considerations from the theory of harmonic maps [3, 11]. One of our goals is to demystify the results of [8, 9] which are based on Fatou's theorem. For a specific elementary equation like (1), involving only two parameters k and w, the method we propose has the following advantage: it provides an easy algorithm of determining the number of solutions for each value of the parameters.

We hope to apply the same method to the more general equation

$$z = \arcsin \frac{k}{\overline{z} + \overline{w}} - \alpha \overline{z}, \quad \alpha \in \mathbf{C}, \tag{2}$$

which describes gravitational lensing by an elliptic object of isothermal density and a shear [7, 8]. It is not clear how to use the arguments based on Fatou's theorem in the case that $\alpha \neq 0$. We will further discuss this at the end of the paper.

We thank D. Khavinson and E. Lundberg for introducing us to the subject and for useful discussions, and we thank C. R. Keeton for very helpful information on the relevance of the model for astronomy.

2. The harmonic map f and its caustic

We are solving the equation $f(z) = w$ where

$$f(z) = z + \overline{g(z)}, \quad g(z) = -\frac{k}{\sin z}, \tag{3}$$

and $z \in D^0 = \{z \in \mathbf{C} : |\operatorname{Re} z| < \pi/2\}$. The Jacobian determinant of f is

$$J(z) = 1 - |g'(\overline{z})|^2 = 1 - |g'(z)|^2.$$

Our map f is smooth and discrete, which means that the preimage of every point is discrete. Such maps are called "light" in [11]. The map preserves the orientation in the region $D^+ = \{z \in \mathbf{C} : J(z) > 0\}$ and reverses the orientation in the complementary region D^-. The common boundary of D^+ and D^- with respect to D^0 is given by the equation

$$|g'(z)| = k \left| \frac{\cos z}{\sin^2 z} \right| = 1.$$

We call this common boundary γ. In the astronomy literature, γ is called the *pre-caustic*, and its image $\Gamma = f(\gamma)$ is called the *caustic*.

For small positive k, the pre-caustic γ consists of a single smooth Jordan curve surrounding the pole at 0. This picture persists for $0 < k < 2$. At $k = 2$, γ bifurcates into four smooth simple curves with endpoints on ∂D^0; see Figures 1-3.

Our count of the number of solutions is based on the Argument Principle for light mappings; see [2, 4, 11]. Let D be a region bounded by finitely many disjoint Jordan curves on the Riemann sphere. We parametrize the boundary curves so that the region stays on the left. Assuming that a smooth map $f : D \to \overline{\mathbf{C}}$, continuous in the closure of D, never takes the values w and ∞ on the boundary ∂D, and that $J(z) \neq 0$ in D, let N and P be the numbers of w-points and poles of f, respectively. Then

$$N - P = \pm I_w(f(\partial D)), \tag{4}$$

where $I_w(f(\partial D))$ is the index (or winding number) of the curves $f(\partial D)$ about w. Thus

$$I_w(f(\partial D)) = \frac{1}{2\pi} \int_{f(\partial D)} \frac{d\zeta}{\zeta - w} = \frac{1}{2\pi i} \int_{\partial D} \frac{df(z)}{f(z) - w},$$

where

$$df = f_z dz + f_{\overline{z}} d\overline{z}.$$

If $J(z) > 0$ in D so that the map preserves the orientation, we choose the plus sign in (4), and if it reverses the orientation, we choose the minus sign.

Our Theorem is an immediate consequence of the following propositions.

Proposition 1. *For every $k > 0$ and every $w \in \mathbf{C}$, we have*

$$|I_w(f(\partial D^-))| \leq 2.$$

Here and in what follows the expression "for every w" means "for every w for which the index is defined". A more careful analysis, which we do not include in this paper, would show that $I_w(f(\partial D^-)) \leq 0$ for all $k > 0$ and all w. This would imply that equation (1) has at least one orientation-reversing solution for all $k > 0$ and $w \in \mathbf{C}$.

Proposition 2. *If $k > 2$, then $|I_w(f(\partial D^+))| \leq 3$ for all $w \in \mathbf{C}$.*

In these propositions, the boundaries ∂D^- and ∂D^+ are understood with respect to the extended plane; if $0 < k < 2$ then $\partial D^- = \gamma$, while if $k > 2$, then ∂D^- consists four components of γ and four vertical intervals. The open set D^+ is always unbounded.

Proposition 2 is actually true for every $k > 0$. This follows from the argument in [8] using Fatou's theorem. We give a proof independent of Fatou's theorem for the case that $k > 2$, which suffices for our purposes.

To derive our Theorem from the propositions, we consider two cases.

If $0 < k < 2$, we apply the Argument Principle and Proposition 1 to D^-, which contains one pole, and obtain that the number of orientation-reversing solutions of (1) is at most 3. Then we apply the Argument Principle to D^+. The boundary of D^+ consists of the curve γ and two vertical lines. The image of the two vertical lines is easy to study and its index about any point in the plane has absolute value at most 1, a fact established in [8]. Thus $|I_w(\partial D^+)| \leq 3$ by Proposition 1. Since there are no poles in D^+, this implies that the number of orientation-preserving solutions of (1) is at most 3. Thus there are at most 6 solutions in this case.

If $k > 2$, the argument is similar. By Proposition 2, the number of orientation-preserving solutions is at most 3, and by Proposition 1, the number of orientation-reversing solutions is at most $2 + 1 = 3$. So our equation has at most 6 solutions in this case as well.

If $k = 2$ the Theorem holds by continuity.

This completes the derivation of our Theorem from Propositions 1 and 2.

Figures 4-7 show the images of the boundaries $f(\partial D^+)$ and $f(\partial D^-)$. The numbers of solutions of (1) are written in the regions complementary to these images. The notation m/n in Figures 4 and 6 means that for w in the indicated region there are m orientation-reversing and n orientation-preserving solutions.

The essential bifurcation occurs at the point $k = 2/\sqrt{3} \approx 1.1546$. In particular, 5 or 6 solutions are only possible for $2/\sqrt{3} < k < k_0 \approx 2.1288$ and the region in the w-plane where the number of solutions is 5 or 6 is rather small. Perhaps this explains the fact that 5 or 6 bright images in a single elliptic lens do not seem to have been observed by astronomers yet.

For some readers these pictures produced by Maple will be sufficiently convincing; these readers may skip the next two sections and pass to the remarks in the end of the paper. The formal proofs which we give below show that the pictures actually represent correctly all essential features of our curves, necessary to determine their indices about every point in the plane. As we mentioned before, some of those features are rather small, and one has to be sure that nothing was missed on a still smaller scale.

3. Study of the caustic $\Gamma = f(\gamma)$ for $k \in (0,2)$

In this section we assume that $0 < k < 2$. The curve γ is given by the equation

$$|g'(z)| = k\left|\frac{\cos z}{\sin^2 z}\right| = 1. \tag{5}$$

We use the local theory of harmonic mappings following the paper by Lyzzaik [11]. We parametrize the curve by

$$t = -\arg g'. \tag{6}$$

This corresponds to the *counter-clockwise* motion on the curve γ. First we determine the cusps of $f(\gamma)$. Let $z(t)$ be the parametrization of γ. The cusps in the image will occur when

$$\frac{d}{dt} f(z(t)) = 0. \tag{7}$$

A simple computation in [11, (2.4), (2.5)] shows that (7) yields

$$\mathrm{Re}(z'(t)e^{-it/2}) = 0. \tag{8}$$

In order to compute the argument of $z'(t)$ we note that the curve γ which is parametrized by $z(t)$ is a level curve of $\log|g'|$. This implies that

$$\arg z'(t) = \arg \mathrm{grad}(\log|g'(z(t))|) - \pi/2 = \arg\left(g'(z(t))/g''(z(t))\right) - \pi/2.$$

So, using (6),

$$\arg\left(z'(t)e^{-it/2}\right) = -\arg g''(z(t)) + (3/2)\arg g'(z(t)) - \pi/2,$$

and condition (8) becomes

$$\frac{g''(z)^2}{g'(z)^3} = \frac{(1+\cos^2 z)^2}{k\cos^3 z} > 0. \tag{9}$$

So to locate the cusps we need to solve two simultaneous equations (5) and (9), that is,

$$k\left|\frac{\cos z}{\sin^2 z}\right| = 1, \quad \frac{(1+\cos^2 z)^2}{\cos^3 z} > 0,$$

where we used that $k > 0$ to drop k from the second equation. Putting $s = \cos z$ we obtain the algebraic equations

$$k^2 \frac{s\overline{s}}{(1-s^2)(1-\overline{s}^2)} = 1 \tag{10}$$

and

$$\frac{(1+s^2)^2}{s^3} - \frac{(1+\overline{s}^2)^2}{\overline{s}^3} = 0. \tag{11}$$

The second equation expresses the condition $(g'')^2/(g')^3 \in \mathbf{R}$; we will later select those solutions that satisfy (9).

Pictures of the algebraic curves defined by (10) and (11) are shown in Figures 8–10 for various values of the parameter k. The part where

$$(1+s^2)^2/s^3 > 0$$

is shown by a bold line.

It is easy to see that there are always 4 real solutions of (10):

$$s = \pm k/2 \pm \sqrt{k^2/4+1}, \tag{12}$$

two of them in the interval $(-1, 1)$ and two outside of this interval.

After simplification and factoring out $s - \overline{s}$ from (11), we obtain

$$s^2 + \overline{s}^2 = 1 - k^2|s|^2 + |s|^4 \tag{13}$$

and

$$s^2 + \overline{s}^2 = |s|^6 - 2|s|^4 - |s|^2. \tag{14}$$

Eliminating $s^2 + \overline{s}^2$ from these two equations, we obtain

$$p(r) = r^3 - 3r^2 + (k^2-1)r - 1 = 0, \quad \text{where} \quad r = |s|^2. \tag{15}$$

The critical values of this polynomial p are

$$(k^2-4)\left(1 \pm \frac{2}{9}\sqrt{12-3k^2}\right),$$

and they are negative for $k \in (0, 2)$. As $p(0) < 0$ we conclude that p has exactly one positive root, which we denote by $r(k)$.

The equation (13) now gives

$$s^2 + \overline{s}^2 = 1 - k^2 r(k) + r^2(k), \quad \text{where} \quad r(k) = |s|^2,$$

and this has solutions if and only if

$$|1 - k^2 r(k) + r^2(k)| \leq 2r(k).$$

Since the inequality $1 - k^2 r + r^2 \geq -2r$ always holds if $k \in (0, 2)$, we only have to consider the second inequality

$$1 - k^2 r(k) + r^2(k) \leq 2r(k).$$

With $q(r) = r^2 - (k^2 + 2)r + 1$ we thus have $p(r(k)) = 0$ and $q(r(k)) \leq 0$. Since $p(r)+q(r) = r^3 - 2r^2 - 3r = r(r+1)(r-3)$ we conclude that $r(k) \leq 3$. Hence $p(3) = 4k^2 - 3 \geq 0$.

So, for $k \in (0, 2)$, the equations (13) and (14) have common solutions if and only if $k \geq 2/\sqrt{3}$. If s is a solution, then so are $-s$, $\overline{s}$ and $-\overline{s}$. For $k > 2/\sqrt{3}$ we find that there are exactly 4 solutions of the system given by (13) and (14), one in each open quadrant. The inequality (9) is satisfied for the solutions in the first and fourth quadrant; cf. Figure 9. Since the cosine is a proper map of degree 2 from D^0 onto the right half-plane, each of these solutions corresponds to 2 solutions for the original variable z.

In addition to these solutions, we always have the 4 real solutions given by (12). Two of these solutions are positive, and these are the ones that satisfy (9). Moreover, one of them is in the interval $(0, 1)$ and one is in the interval $(1, \infty)$. In the original variable z the first one corresponds to 2 real solutions while the second one corresponds to 2 solutions on the imaginary axis.

Altogether we thus conclude that there are 4 or 8 points z on γ such that $f(z)$ is a cusp of Γ. Of the 4 points corresponding to the real solutions (12), there is one on each coordinate semi-axis. We label these points as z_1, z_2, z_3, z_4 where $z_1 > 0$, $z_2 = ic, c > 0$, $z_3 = -z_1$ and $z_4 = -z_2$. For $k > 2/\sqrt{3}$ there are 4 further solutions w_1, w_2, w_3, w_4 which we label such that w_j is in the j–th quadrant; see Figures 1 and 2.

Now we find the position of the cusps $f(z_j)$. Since $0 < z_1 < \pi/2$, we deduce from (5) that $k \cos z_1 / \sin^2 z_1 = 1$ and hence $k/\sin z_1 = \tan z_1$. Thus

$$f(z_1) = z_1 - k/\sin z_1 = z_1 - \tan z_1 < 0.$$

Similarly we figure out where the other three cusps $f(z_j)$ are located and find that

$$f(z_1) < 0, \quad f(z_3) > 0, \quad f(z_2)/i > 0, \quad \text{and} \quad f(z_3)/i < 0. \tag{16}$$

Thus we obtain the following result.

Lemma 1. *The caustic Γ has 4 cusps if $0 < k < 2/\sqrt{3}$ and 8 cusps if $2/\sqrt{3} < k < 2$. For every $k \in (0,1)$ there are 4 cusps on the coordinate axes located as in* (16)*. For $2/\sqrt{3} < k < 2$ there are 4 additional cusps $f(w_j)$.*

The tangent vectors at the cusps are horizontal on the real line and vertical on the imaginary line. This follows from the symmetry of Γ with respect to reflections in the coordinate axes.

Lemma 2. *The tangent vector to $\Gamma = f(\gamma)$ is never vertical, except on the imaginary axis, and never horizontal, except on the real axis.*

Proof. By (8) this tangent vector is colinear to $\pm e^{it/2}$, where t is the parameter defined in (6). As z describes the curve γ once counter-clockwise, e^{it} describes the unit circle twice counter-clockwise, and we know that at the intersections of γ with the coordinate axes correspond to the intersections of the unit circle with the real line. It follows that the total change of $t/2$ on each smooth arc of Γ is at most $\pi/2$ which proves the Lemma.

A smooth curve will be called *convex* if its tangent vector turns to the left all the time. In other words, $\arg \zeta'(t)$ strictly increases, where $\zeta(t)$ is the parametrization of the curve. We will use the following fact:

Lemma 3. [11, Theorem 2.3] *Each smooth piece of Γ, parametrized as explained above as $\zeta(t) = f(z(t))$ between the cusps, is convex. At the cusps the argument of the tangent vector jumps by π.*

Lemmas 1-3 are sufficient for the proof of all properties of Γ we need.

Proof of Proposition 1 for $0 < k < 2$. First we consider the case that $0 < k < 2/\sqrt{3}$. The curve Γ has 4 cusps, one on each coordinate semi-axis. Begin tracing Γ from the cusp $f(z_3)$ on the positive semi-axis, where its tangent has direction π. As the argument of the tangent increases and can never reach $3\pi/2$, this arc ends at the cusp on the negative ray of the imaginary axis. Both $\operatorname{Re}\zeta$ and $\operatorname{Im}\zeta$ are monotone on the arc, so it belongs to the 4-th quadrant. The other three smooth arcs of Γ are obtained by symmetry with respect to both axes. Thus Γ is a simple Jordan curve, as shown in Figure 4, and the index of Γ about any point w can be only 0 or ± 1.

Now we consider the case that $2/\sqrt{3} < k < 2$. Let γ_0 be the arc of γ from z_3 to z_4, and put $\Gamma_0 = f(\gamma_0)$. With $w_3 \in \gamma_0$ defined as above, the points $f(z_3)$, $f(w_3)$ and $f(z_4)$ are consecutive cusps on Γ_0. Denote by Γ_1 the arc of Γ_0 from $f(z_3)$ to $f(w_3)$, and by Γ_2 the arc from $f(w_3)$ to $f(z_4)$. Then the tangent to Γ_1 at $f(z_3)$ has argument π and increases but never reaches $3\pi/2$ on Γ_1. It follows that $\operatorname{Im}\zeta$ *decreases* on Γ_1.

Let $a \in (\pi, 3\pi/2)$ be the argument of Γ_1 at $f(w_3)$. The next arc Γ_2 of Γ_0 begins at $f(w_3)$ with the argument of the tangent $a - \pi \in (0, \pi/2)$ and then the argument of the tangent increases but reaches the value $\pi/2$ only at the endpoint $f(z_4)$ on the negative imaginary axis. It follows that $\operatorname{Im}\zeta$ *increases* on Γ_2. So Γ_0 intersects every horizontal line at most twice.

Another conclusion from these arguments is that Γ_0 belongs to the lower halfplane, except for the point $f(z_3)$. Indeed, $\operatorname{Im}\zeta$ decreases on Γ_1 from 0 to some negative value, and then $\operatorname{Im}\zeta$ increases on Γ_2 and ends with a negative value.

Let Γ_3 be the curve obtained by reflecting Γ_0 in the imaginary axis and reversing the orientation. Then the sum $\Gamma_4 = \Gamma_0 + \Gamma_3$ intersects all horizontal lines at most 4 times, and does not intersect horizontal lines in the upper half-plane. Let Γ_5 be the curve obtained by reflecting Γ_4 in the real axis and changing the orientation. Then the sum $\Gamma_6 = \Gamma_4 + \Gamma_5$ intersects every horizontal line at most 4 times. On the other hand, $\Gamma_6 = \Gamma$, and we conclude that the index of Γ has absolute value at most 2.

4. The case $k > 2$

Again we begin with a study of cusps of Γ. The following analysis, similar to that in the previous section, will show that that for $k > 2$, the caustic Γ has exactly four cusps, one on each coordinate semi-axis.

The cubic (15) has no real critical values, so it has only one root $r(k)$. This root defines 4 solutions of the system (13) and (14); see Figure 10. However, it is easy to check that two of these roots correspond to negative values of $(1+s^2)^2/s^3$ and two other roots have preimages under cosine which are outside of the region D^0. This proves our claim.

We denote the cusps by z_1, z_2, z_3, z_4, where $z_1 > 0, z_2/i > 0, z_3 = -z_1$ and $z_4 = -z_2$.

Proof of Proposition 2. The open set D^+ consists of 4 regions D_j^+, $1 \le j \le 4$, where D_1^+ intersects the positive real axis, D_2^+ intersects the negative real axis, D_3^+ intersects the positive imaginary axis and D_4^+ intersects the negative imaginary axis.

Let $L = [\pi/2 - it_0, \pi/2 + it_0]$ be the vertical segment of ∂D_1^+. It is easy to see that t_0 is defined by

$$\sinh t_0 = k/2 - \sqrt{k^2/4 - 1} < 1 \quad \text{for} \quad k > 2.$$

An easy computation shows that $f(L)$ is the graph of a convex function $x = \phi(y)$. Indeed, we have

$$\phi(y) = \pi/2 - k/\cosh y,$$

and this is convex for $\sinh y \in [-1, 1]$.

So $f(\partial D_1^+)$ consists of three convex curves. Two of them are symmetric to each other with respect to the real line and meet at a cusp on the real line. As the variation of the argument of the tangent to each of these two curves is less than $\pi/2$, the union of these curves is a graph of a function $x = x_1(y)$. The third curve is a graph of a function $x = x_2(y)$.

It follows that every horizontal line intersects $f(\partial D_1^+)$ at most twice, so the index of $f(\partial D_1^+)$ with respect to every point in the plane has absolute value at most 1.

The same argument works for ∂D_2^+. So $|I_w(f(\partial D_j^+))| \le 1$ for every w and $j = 1, 2$.

Concerning the curves $f(\partial D_j^+)$ for $j = 3, 4$, each of them intersects every horizontal line at most twice. So the index of these curves has absolute value at most 1.

Finally the curves $f(\partial D_j^+), j = 3, 4$ do not intersect because one of them belongs to the upper half-plane and another belongs to the lower half-plane.

Thus $f(\partial D^+)$ is the union of 4 curves, each of them has index of absolute value at most 1 with respect to any point, and two of these curves belong to the complementary half-planes. Proposition 2 follows.

It remains to prove Proposition 1 for $k > 2$. We use the same method as in the previous section.

Proof of Proposition 1 for $k > 2$. Let v_1 and v_2 be the endpoints of γ on the line $\operatorname{Re} z = -\pi/2$ such that $\operatorname{Im} v_2 < \operatorname{Im} v_1 < 0$; see Figure 3. Let γ_0 be the arc of ∂D^- from z_3 to z_4. We break γ_0 into three arcs:

γ_1 from z_3 to v_1,
γ_2 from v_1 to v_2, and
γ_3 from v_2 to z_4.

The images of γ_j, $0 \leq j \leq 3$, under f are denoted by Γ_j.

The imaginary part $\operatorname{Im}\zeta$ is decreasing on Γ_1 for the same reasons as in the previous section: Γ_1 is a convex curve which begins at $f(z_3)$ with horizontal tangent, and the argument of the tangent increases on Γ_1 but never reaches a vertical direction.

The imaginary part $\operatorname{Im}\zeta$ is decreasing on Γ_2 because

$$\operatorname{Im} f(-\pi/2 - it) = -it.$$

Finally, $\operatorname{Im}\zeta$ is monotone on Γ_3 because Γ_3 ends on the imaginary axis with the vertical tangent, and this tangent never else has a vertical direction.

We conclude that Γ_0 intersects every horizontal line at most twice, and Γ_0 belongs to the lower half-plane. As ∂D^- is the union of 4 curves obtained from Γ_0 by reflections in the axes, we conclude that ∂D^- intersects each horizontal line at most 4 times, so its index does not exceed 2 by absolute value. This completes the proof.

5. Remarks

1. To demonstrate that any number of solutions of equation (1) between 1 and 6 can actually occur, we pick appropriate values of w from our Figures 4-7, or similar figures for other values of k.

For example, $k = 1.92$ and $w = 0.67i$ gives 6 solutions $1.5363458i$, $-0.9885626i$, $\pm 1.2603941 + 0.97328106i$, $\pm 1.4617539 + 0.77388761i$,

2. The region with 6 solutions exists for $k \in (2/\sqrt{3}, k_0)$, where $k_0 \approx 2.1288$ is determined from the equation

$$16k^3 - 4\pi^2 k^2 - \pi^4 = 0.$$

This is the condition that $\operatorname{Re} f(v_1) = 0$, where v_1 was defined in the proof of Proposition 1 for $k > 2$ in Section 4.

3. Equation (2) can be rewritten in a form similar to equation (1). Suppose that $\alpha \notin \{1, -1\}$ and put $u = z - \alpha\overline{z}$. Then $\overline{z} = (\overline{\alpha}u + \overline{u})/(1 - |\alpha|^2)$,

and the equation becomes

$$u = \arcsin \frac{k_1}{\overline{\alpha} u + \overline{u} + \overline{w_1}},$$

where $k_1 = k(1 - |\alpha|^2)$, and $w_1 = w(1 - |\alpha|^2)$. Now we take the sine on both sides and conjugate to obtain

$$u = \frac{k_1}{\sin \overline{u}} - \alpha \overline{u} - w_1. \tag{17}$$

This can be considered a perturbation of the equation (1) by the term $\alpha\overline{u}$. Orientation-preserving solutions of equation (17) are attracting fixed points of the anti-analytic entire function $h(u) = k_1/\sin \overline{u} - \alpha\overline{u} - w_1$. Fatou's theorem says that every attracting fixed point attracts a trajectory of a singular value. If $\alpha = 0$ the function has 3 singular values, so there are at most 3 attracting fixed points. This is the crucial part of the argument in [8]. For $\alpha \neq 0$, the function h has infinitely many critical values, so the dynamical proof breaks down at this point.

In the case $\alpha = 0$ considered in this paper, h indeed can have 3 distinct attracting points. This happens for those values of the parameters k and w for which we have 3 orientation-preserving solutions, for instance for $(k, w) = (1.92, 0.67i)$ as in in the example above. Figure 11 shows the partition of the plane into three domains of attraction of the fixed points: the attracting basin of $1.536345814i$ is shown in white, that of $1.46175395 + 0.7738876153i$ in black and that of $-1.46175395 + 0.7738876153i$ in grey. The Julia set of this function has zero area [6, Theorem 3], and it is not visible in the pictures.

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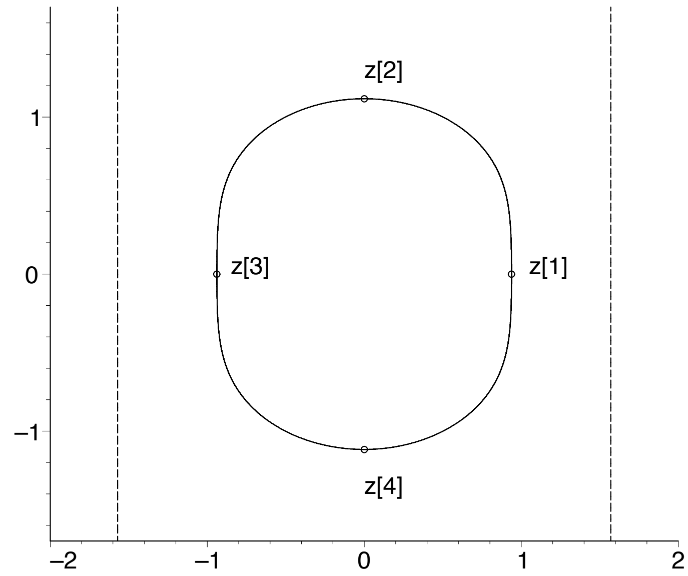

Figure 1: Pre-caustic for $k = 1.1$.

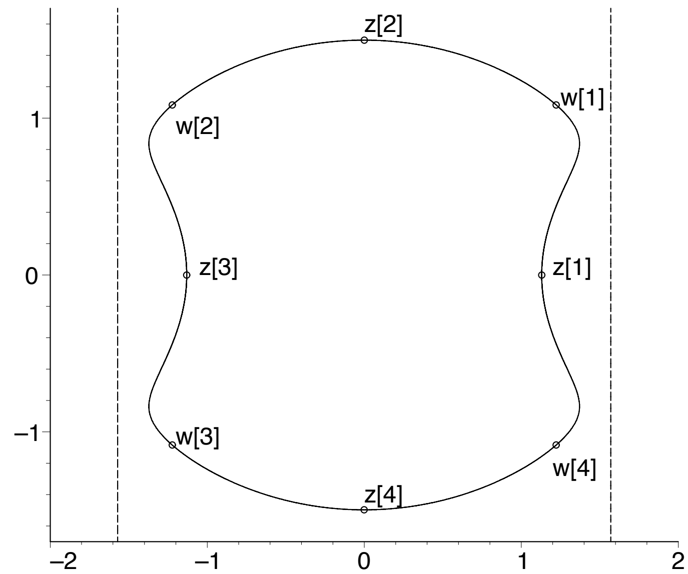

Figure 2: Pre-caustic for $k = 1.92$.

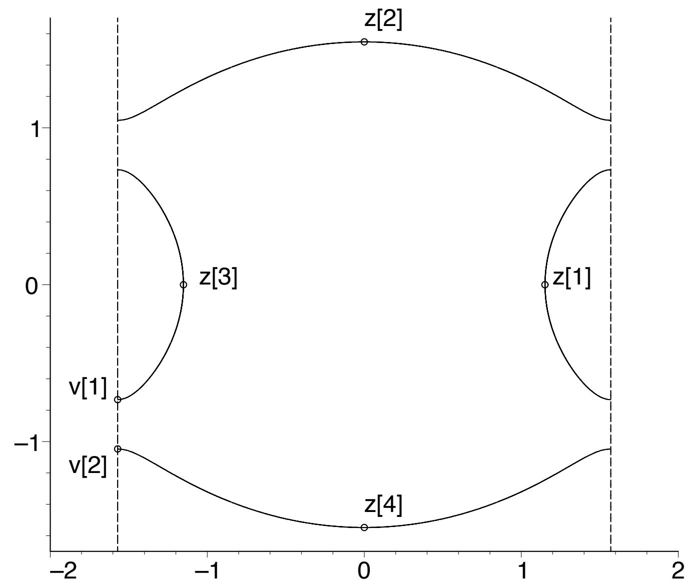

Figure 3: Pre-caustic for $k = 2.05$.

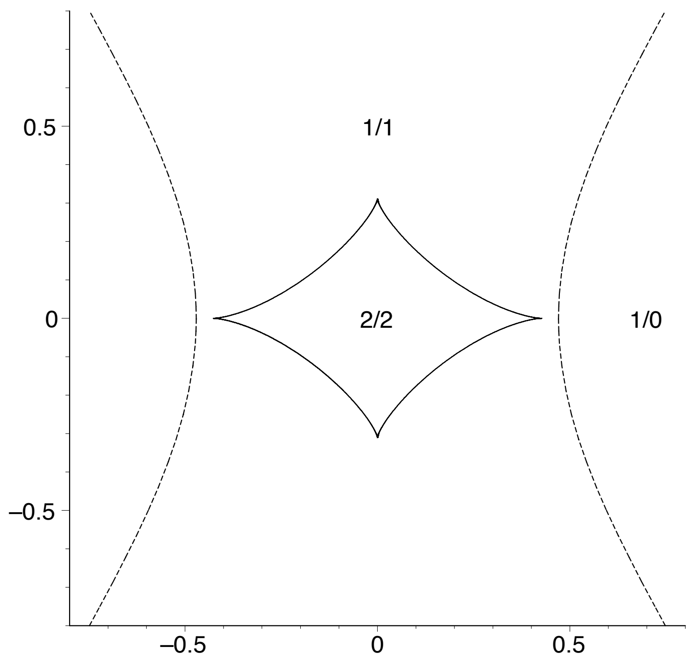

Figure 4: The caustic Γ for $k = 1.1$. The image of the boundary $f(\partial D^0)$ is shown in dotted lines.

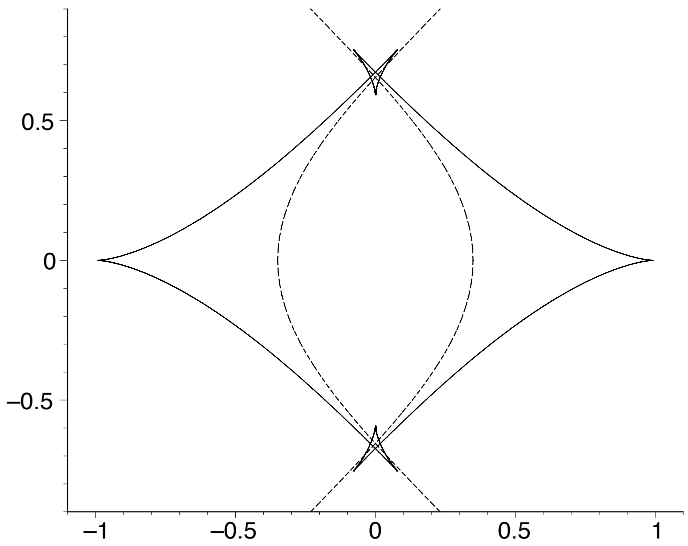

Figure 5: The caustic and the image of the boundary of D^0 for $k = 1.92$.

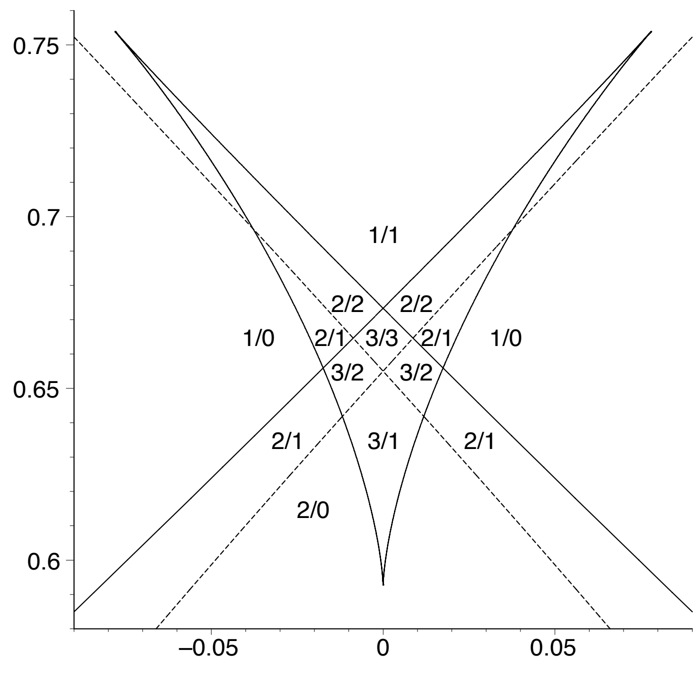

Figure 6: Magnification of detail from Figure 5.

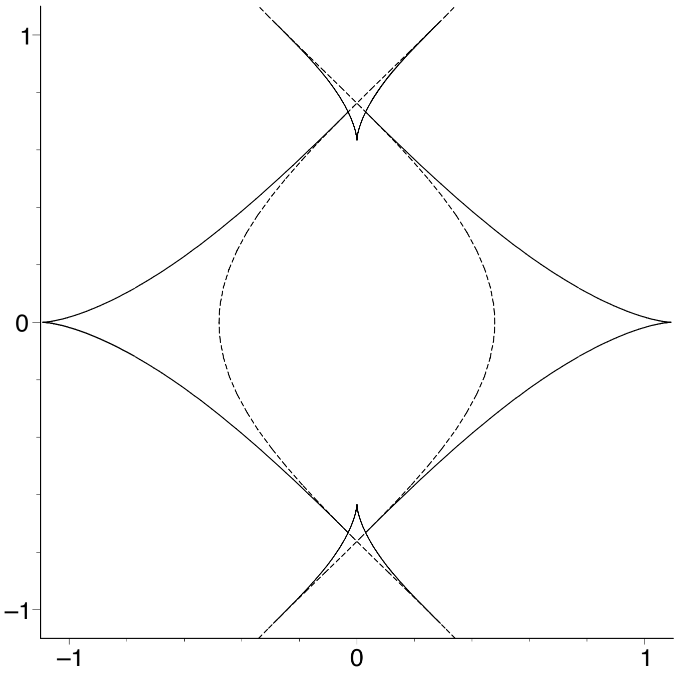

Figure 7: The caustic and the image of the boundary of D^0 for $k = 2.05$.

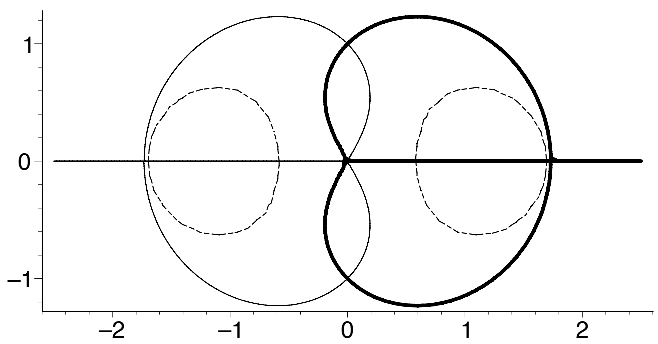

Figure 8: Curves (10) in dotted lines and (11) in solid lines for $k = 1.1$.

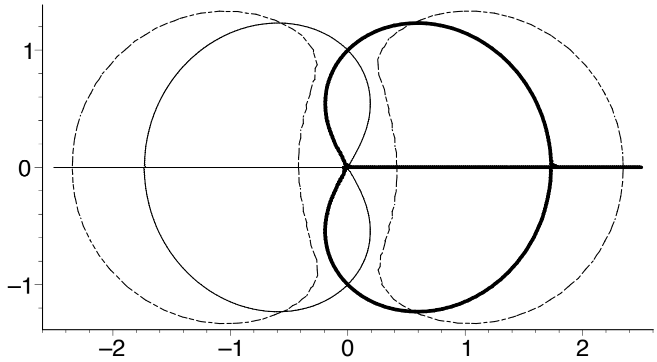

Figure 9: Curves (10) in dotted lines and (11) in solid lines for $k = 1.92$.

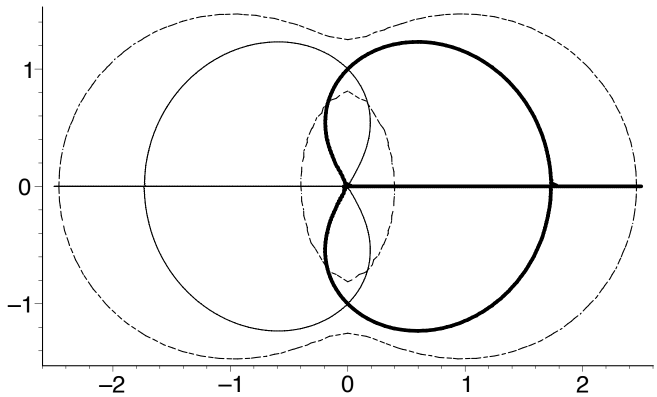

Figure 10: Curves (10) in dotted lines and (11) in solid lines for $k = 2.05$.

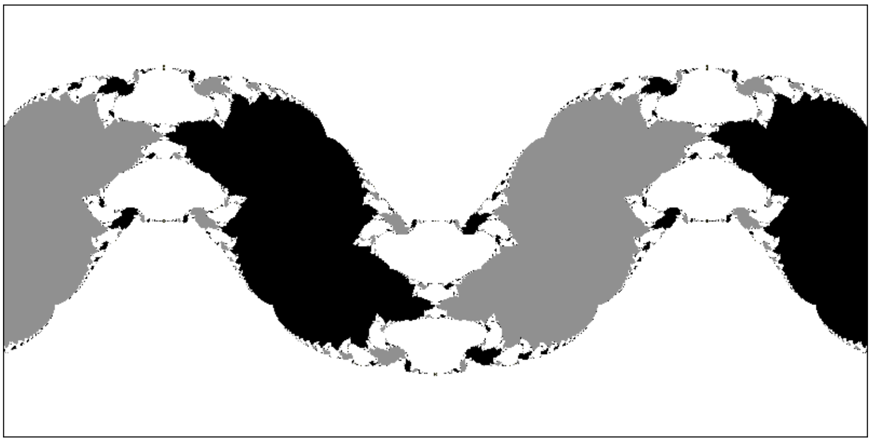

Figure 11: Basins of attraction of $h(z) = 1.92/\sin\overline{z} + 0.67i$.