3.1 Here is another way of deriving the formula for general solution of a linear differential equation

$$y' + p(x)y + q(x) = 0. \tag{1}$$

Let us first consider a *homogeneous linear equation*

$$y' + p(x)y = 0 \tag{2}$$

(do not confuse it with a general homogeneous (non-linear) equation!) (2) is separable, and its general solution is

$$y = C \exp\left(-\int p(x)dx\right) =: Cy_0(x), \tag{3}$$

where C is an arbitrary constant. Now we look for a solution of the non-homogeneous equation (1) in the form

$$y(x) = C(x)y_0(x), \tag{4}$$

where this time $C(x)$ in still unknown function of x. This is called *"method of variation of constant"*. After the general solution of the homogeneous linear equation is found, we let arbitrary constants in it be functions of the independent variable, and plug this to original equation to find these functions.

Substituting (4) to the original equation (1), we obtain

$$C'(x)y_0(x) + C(x)y_0'(x) + p(x)C(x)y_0(x) + q(x) = 0,$$

but the second and third summands add up to zero because y_0 satisfies the homogeneous equation (2). Thus

$$C'(x)y_0(x) + q(x) = 0,$$

which gives

$$C(x) = -\int \frac{q(x)}{y_0(x)}dx.$$

Combining this with (4) and plugging the expression (3) for y_0, we finally obtain

$$y(x) = e^{-\int p(x)dx}\left\{-\int q(x)e^{\int p(x)dx}dx\right\}$$