

# Some constants coming from the work of Littlewood

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1. Let

$$\phi(n) = \sup_{\deg p=n} \int_{|z|<1} \frac{|p'|}{1+|p|^2} dm,$$

where the sup is taken over all polynomials of degree  $n$  and  $dm$  stands for the area element. Littlewood [10] gives several equivalent definitions of  $\phi(n)$ . He observed that  $\phi(n) \leq \pi\sqrt{n}$ . Let

$$\alpha = \limsup_{n \rightarrow \infty} \frac{\log \phi(n)}{\log n}.$$

Littlewood's conjecture that  $\alpha < 1/2$  was proved by Lewis and Wu [9], who showed that in fact  $\alpha < 2^{-1} - 2^{-264}$ . These authors used an approach from [6, 7] where a weaker result was obtained. Early lower estimates of  $\alpha$  are due to Paley and Hayman, but only in [4] it was proved that  $\alpha > 0$ . The rigorous numerical estimate  $\alpha > 10^{-5}$  is due to Baker and Stallard [1]. Using a computer, Kraetzer [8] showed that in fact the method proposed in [4] gives  $\alpha > 0.242$ . This confirms an earlier computation of Carleson and Jones [2].

2. Let  $E$  be a regular compact subset of the plane and  $G$  the Green function of  $\overline{\mathbb{C}} \setminus K$  with the pole at infinity. For every  $\epsilon > 0$ , let  $l(\epsilon)$  be the length of the level curve  $\{z : G(z) = \epsilon\}$ . Put

$$\beta_E = \limsup_{\epsilon \rightarrow 0} \frac{\log l(\epsilon)}{-\log \epsilon},$$

and

$$\beta = \sup \beta_E$$

over all *connected* compact sets. It is known that  $.17 < \beta < .49$ . The upper bound is due to Clunie and Pommerenke [3] and the lower bound to Pommerenke [11]. The problem of estimating  $\beta$  comes from Littlewood's work on univalent functions. He proved that  $\beta > 0$ . Computer experiments of Carleson and Jones [2] and Kraetzer [8] indicate that  $\beta > .242$ , which seems to confirm the conjecture Carleson and Jones that  $\beta = 1/4$ .

One can ask the same question about  $\sup \beta_E$  over all regular compact sets, not necessarily connected. I conjecture that the sup is attained on connected sets. Paper [5] gives some evidence of this.

3. All lower estimates in [1, 2, 4, 8] are all based on iteration theory, more precisely on "thermodynamic formalism". Let  $p_c(z) = z^2 + c$  be a hyperbolic polynomial ("hyperbolic" means that the trajectory of 0 under the iterates of  $p_c$  tends to an attracting cycle, possibly to infinity). Denote the  $n$ -th iterate of  $p_c$  by  $p_c^n$ . The number

$$P_c = \lim_{n \rightarrow \infty} \frac{1}{n} \log \sum_{z: p_c^n(z)=1} |(p_c^n)'|^{-1}$$

is called the *pressure, corresponding to*  $|(p_c^n)'|^{-1}$ . The existence of the limit and positivity of  $P_c$  for  $c \neq 0$  was shown by Ruelle [12] who also proved that  $c \mapsto P_c$  is a real analytic function on every component of the set of parameters  $c$  where  $p_c$  is hyperbolic. It can be shown [4] that  $\alpha \geq P_c / \log 2$  for all such  $c$ . Similarly, it is shown in [2] that  $\beta \geq P_c / \log 2$  for all  $c$  such that the trajectory of 0 tends to a finite attracting cycle.

4. I do not know of any evidence in favor or against the Carleson and Jones conjecture, except the computer experiments mentioned above. But there are several questions which seem to be easier:

- A. Is there any connection between  $\alpha$  and  $\beta$ ? Is it true that  $\alpha = \beta$ ?
- B. How to obtain better estimates of  $\alpha$ ,  $\beta$  and  $\sup_c P_c$ , even with the help of a computer?
- C. Are the polynomials  $P_c^n$  extremal or nearly extremal for  $\alpha$ ?
- D. For which  $c$  is the pressure  $P_c$  close to its supremum?

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