1. Suppose $A \subset \mathbb{R}^p$ and $B \subset \mathbb{R}^q$ and consider the product $$A \times B = \{(x, y) \in \mathbb{R}^{p+q} : x \in A, y \in B\}.$$ (a) Show that if $A$ and $B$ are open in $\mathbb{R}^p$ and $\mathbb{R}^q$, respectively, then $A \times B$ is open in $\mathbb{R}^{p+q}$.

(b) Similarly, show that if both $A$ and $B$ are closed, then $A \times B$ is closed.

2. Let $(x_n)_{n=1}^\infty$ be a sequence of points in $\mathbb{R}^p$ with the property that there exists a real number $r$, $0 < r < 1$, and an integer $N_0$ such that $$\|x_{n+1} - x_n\| \leq r \|x_n - x_{n-1}\|$$ for all $n \geq N_0$. Then prove $(x_n)_{n=1}^\infty$ is a convergent sequence.

3. Let $(x_n)_{n=1}^\infty$ be a sequence in a compact set $K \subset \mathbb{R}^p$ that is not convergent. Show that there are two subsequences of this sequence that are convergent to different limit points.

4. We say that $x$ is a limit point of the sequence $(x_n)$ if $x = \lim_{k \to \infty} x_{n_k}$ for a certain subsequence $(x_{n_k})$. Construct a compact set of real numbers whose limit points form a countable set.

5. Prove that if $A$ and $B$ are disjoint closed sets in $\mathbb{R}^n$, then $A \cup B$ is disconnected.

6. Let $(a_n)$ be a sequence of real numbers of such that $a = \lim_{n \to \infty} a_n$. Show that the sequence of arithmetic means $$\alpha_n = \frac{a_1 + a_2 + \cdots + a_n}{n}$$ is also convergent and $\lim_{n \to \infty} \alpha_n = a$.

7. Find the upper and lower limits of the sequence $(x_n)$ defined by $$x_1 = 0, \quad x_{2m} = \frac{x_{2m-1}}{2}, \quad x_{2m+1} = \frac{1}{2} + x_{2m}.$$ 

8. Suppose that $(x_n)$ and $(y_n)$ are Cauchy sequences in $\mathbb{R}^p$. Show that the numeric sequence $d_n = \|x_n - y_n\|$ is convergent.