MA440 REAL ANALYSIS (HONORS)

MIDTERM EXAM 2 PRACTICE PROBLEMS

1. Suppose that $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is a continuous function such that $\lim_{|x| \to \infty} f(x) = 0$, i.e., for all $\epsilon > 0$, there is an $N$ so that $|f(x)| < \epsilon$ for all $x$ with $|x| > N$. Show that $f$ is uniformly continuous on $\mathbb{R}^n$.

2. Let $f$ be a continuous function on $\mathbb{R}$ to $\mathbb{R}$ which does not take on any of its values twice. It it true that $f$ must either be strictly increasing (in the sense that if $x' < x''$ then $f(x') < f(x'')$) or strictly decreasing?

3. Let $f$ be defined for all real $x$, and suppose that $|f(x) - f(y)| \leq (x - y)^2$ for all real $x$ and $y$. Prove that $f$ is constant.

   *Hint:* Show that $f$ is differentiable and find $f'$.

4. Let $F$ be the Cantor set. Let $f$ be a bounded real function on $[0, 1]$ which is continuous at every point outside $F$. Prove that $f$ is Riemann integrable on $[0, 1]$.

   *Hint:* $F$ can be covered by finitely many segments whose total length can be made as small as desired.

5. If $f : [0, 1] \rightarrow [0, \infty)$ is increasing and $f \left( \frac{1}{2} \right) > 1$, show that

   $$\int_0^1 f(x)dx > \frac{1}{2}.$$

6. Show that $f_n(x) = n \sin(x/n)$ converges uniformly on $[-a, a]$ for any finite $a > 0$ but does not converge uniformly on $\mathbb{R}$.

7. Show that a monotone function $f : [a, b] \rightarrow \mathbb{R}$ is continuous if and only if its image $f([a, b])$ is an interval.

8. Suppose
   - $f$ is continuous for $x \geq 0$,
   - $f'(x)$ exists for $x > 0$,
   - $f(0) = 0$,
   - $f'$ is monotonically increasing.

   Put

   $$g(x) = \frac{f(x)}{x}, \quad x > 0$$

   and prove that $g$ is monotonically increasing.