PROJECT SUMMARY

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Real meromorphic functions

Submitting Organization: Purdue Research Foundation

The proposer intends to continue his study of the distribution of roots and critical points of real meromorphic functions, using mainly the methods of geometric theory of meromorphic functions, the approach which already brought significant results in his previous research.

He plans to concentrate on the following specific problems:

1. The study of the Wronski map, both in the real and complex domain, and of the related pole assignment map.

2. The study of the distribution of roots of successive derivatives of real entire functions.

3. Further investigation of the relation between the rate of oscillation of real functions and their spectral properties.

4. The study of existence and uniqueness of metrics of constant positive curvature with conic singularities on compact surfaces.

Proposed research will expand our knowledge of the distribution with respect to the real axis of solutions of real algebraic and transcendental equations that occur in geometry and analysis.

As a broader impact, these results will have applications in real enumerative geometry, control theory of linear systems by static output feedback and possibly in the mathematical theory of signal processing. In real enumerative geometry, new lower estimates of the numbers of real solutions of certain enumerative problems are expected. In control theory, new results are anticipated on the possibility of pole assignment for a linear sytem by real, static output feedback.

RESULTS FROM PRIOR NSF SUPPORT

The research was titled "Geometric Theory of Meromorphic Functions", and was supported by NSF grant DMS–0100512 (Funding period: June 1, 2001–May 31 2006). The amount of support was $254,147. The results of this research are contained in the 18 papers listed below (15 published and 3 accepted for publication). In addition, three preprints are posted on the web, but not submitted to journals yet.

1. (with A. Gabrielov) Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, *Ann of Math.,* 155 (2002), 105-129.

We prove the first non-trivial case ($p = 2$) of the so-called Shapiro Conjecture, from real enumerative algebraic geometry. This conjecture will be discussed in the next section "Project Description,." The case proven in the paper can be restated as a result about rational functions of one variable: *if all critical points of a rational function f belong to a circle C on the Riemann sphere, then $f(C)$ is a subset of some circle.*

The Shapiro Conjecture has applications to the real Schubert Calculus [61], geometry of real algebraic curves [33], and control theory [53, 62]. General surveys on this conjecture are [61, 62]. Our proof combines methods from several areas: geometric function theory, algebraic geometry, combinatorics and topology.

2. (with A. Gabrielov) Counterexamples to pole placement by static output feedback, *Linear Algebra and Appl.,* 351-352 (2002) 211-218.
3. (with A. Gabrielov) Pole placement by static output feedback for generic linear systems, *SIAM J. on Control and Opt.,* 41, 1 (2002) 303–312.

In these two papers, we apply the established cases of the Shapiro Conjecture to control theory of linear systems by static output feedback. We show that one of the main problems of this theory, the pole placement problem in the critical case [9], is unsolvable by real static output feedback for an open set of data.

4. (with A. Gabrielov) Wronski map and Grassmannians of real, codimension two subspaces, *Computational Methods and Function Theory*, 1 (2001) 1-25.
5. (with A. Gabrielov) Degrees of real Wronski maps, *Discrete and Computational Geometry,* 28 (2002) 331–347.

In these two papers we study the Wronski map that sends a vector of real polynomials into their Wronski determinant. The main result is a computation of the topological degree of this map. This result has applications to problems of real enumerative geometry: it gives non-trivial lower estimates of the numbers of real solutions of Schubert-type geometric enumerative problems. This is one of the first results of this type in real algebraic geometry (Only one other result of this type is known to the proposer: a non-trivial lower estimate of the number of real plane rational curves of given degree passing through the given points; this is due to Degtyarev and Kharlamov for degree 3, and to Welschinger, Mikhalkin, Itenberg, Kharlamov and Shustin in general [30].)

6. (with A. Gabrielov, M. Shapiro and A. Vainshtein) Rational functions and real Schubert calculus, math.AG/0407408. Accepted in *Proc. AMS.*

This is a further development of paper 1, further exploring the connection between the geometry of rational functions of one variable and the Schubert calculus. We state a generalization of the Shapiro Conjecture, and prove it in the first non-trivial case $p = 2$. The result gives new lower estimates in the Schubert calculus of flags, and an improvement of the recent theorem on the number of non-equivalent rational functions with prescribed critical points [57, 56]. Our paper was inspired by the massive numerical experiments in [55], and we rigorously prove some conjectures made there.

7. Value distribution and potential theory, *Proceedings of the International Congress of Mathematicians,* vol. 2, p. 681-690, Higher Education Press, Beijing, 2002.

This is the invited 45 min talk on ICM in Beijing where a survey was given of the Potential-theoretic approach to the value distribution theory of meromorphic functions, holomorphic curves and quasiregular mappings. This approach was developed by the PI in 1990-s, partially in co-operation with M. Sodin and J. Lewis. Most of this research was funded by NSF grants of the proposer.

8. Geometric theory of meromorphic functions, in the book: *In the Tradition of Ahlfors-Bers, III,* (Contemp. math., 355) AMS, Providence, RI, 2004, pp. 221-230.

A survey of the area to which my current NSF-sponsored research belongs.

9. (with W. Bergweiler and J. Langley) Real entire functions of infinite order and a conjecture of Wiman, *Geometric and Functional Analysis (GAFA),* 13, 5 (2003), 975-991.

This paper completes the long development, proving in full generality the following conjecture of Wiman (1911): if f is a real entire function such that ff'' has only real zeros then f belongs to the Laguerre–Pólya class, that is f is a limit of polynomials with real zeros. We build on the two major previous steps towards this result, [39] and [58], and add some new arguments from geometric theory of meromorphic functions.

10. (with W. Bergweiler and J. Langley) Zeros of differential polynomials in real meromorphic functions, *Proc. Edinburgh Math. Soc.* (2005) 48 1-15.

This is a further development of the methods introduced in the paper 9, combined with arguments from holomorphic dynamics. We obtain "real" versions of the classical results of Hayman [26] on distribution of zeros of differential polynomials.

11. (with D. Novikov) Oscillation of functions with a spectral gap, *Proc. Nat. Acad. Sci.,* 101, 16 (2004), 5872-5873.
12. (with D. Novikov) Oscillation of Fourier integrals with a spectral gap, *J. de Math. Pures et Appl.,* 8, 3 (2004), 313-365.

We prove a conjecture of B. Logan [40], that all real functions of restricted growth with a symmetric spectral gap $(-a, a)$ oscillate fast, namely the lower density of their sign changes is at least a/π. Such functions are encountered in communication engineering where they are called "high-pass signals". We also show that some natural generalizations of this conjecture for faster growing functions fails, and find an exact growth condition for which the conjecture is true.

13. Transcendental meromorphic functions with three singular values, *Illinois J. Math.,* 48 (2004), 701-709.

It was recently discovered by J. Langley that meromorphic functions with finitely many singular values cannot have arbitrary rate of growth. A precise estimate of growth from below is given in this paper. This involves an extremal problem whose solution has hexagonal symmetry. This type of symmetry is expected in solutions of several unsolved problems of geometric theory of meromorphic functions, see, for example, [7]. Recently my former student Merenkov showed that there are no growth restrictions from above in this problem.

14. Metrics of positive curvature with conic singularities on the sphere, *Proc. AMS,* 132 (2004), 11, 3349–3355.

The problem of existence of a metric of constant curvature on a compact surface with prescribed conic singularities has a long history going back to Picard and Poincaré. It is now completely solved for the case of non-positive curvature: the only necessary restriction in this case is the Gauss–Bonnet formula, and the solution is essentially unique [64]. In my paper, the simplest unsolved case is treated, the case of the metric of positive curvature on the sphere with three singularities. A complete solution for this case is given. It shows that there are complicated restrictions besides the Gauss–Bonnet formula. Also the first examples of non-uniqueness are given, based on the PI's work on the Shapiro Conjecture (paper 1). The analytic aspect of the problem is solving the equation

$$\Delta u = -e^u \tag{1}$$

on the punctured sphere with prescribed singularities at the punctures. The problem is of interest in several areas of physics, see, for example [52, 5].

15. Critical values of generating functions of totally positive sequences, *Matematicheskaya fizika, analiz, geometriya,* 11 (2004), 4, 1-13. (Math. Physics, Analysis and Geometry (MAG)).

This is an attempt to extend some results of paper 1 to the transcendental case and simultaneously to generalize the results of G. McLane [41] and E. Vinberg on geometric characterization of the Laguerre–Polya–type classes of entire functions to similar classes of meromorphic functions.

16. (with A. Baernstein II, A. Fryntov and A. Solynin) Sharp estimates for hyperbolic metrics and covering theorems of Landau type, *Ann. Acad. Sci. Fenn., Math.* 30 (2005) 113-133.

We obtain some new covering theorems of Landau type, with sharp constants. This involves solving extremal problems for solutions of the nonlinear equation $\Delta u = e^u$ in variable domains. Ahlfors' method of ultrahyperbolic metrics is the main tool.

17. (with S. Merenkov) Nevanlinna functions with real zeros,
math.CV/0405196. Accepted in *Illinois J. Math.*

This paper answers a question of S. Hellerstein and J. Rossi [8] about entire solutions of the differential equation $w'' + Pw = 0$, where P is a polynomial. We show that there exist polynomials P of all degrees d, such that the equation has solutions w with only real zeros. Previously this was known only for $d = 0, 1, 2$ and 4, and recently K. Shin [59] independently established the result for $d = 3$.

18. (with W. Bergweiler) Meromorphic functions with two completely invariant domains, math.CV/310495. Accepted for the *Memorial volume of I. N. Baker,* 2005.

We give a complete description of dynamics of critically finite meromorphic functions with two completely invariant domains, generalizing the results of Fatou for rational functions. In particular, we prove that the Julia set of such function has to be a Jordan curve on the Riemann sphere, as conjectured by Baker.

Preprints.

1. A Toda lattice in dimension 2 and Nevanlinna theory, math.AP/0411151

This is the first attempt of the proposer to learn the higher-dimensional generalization of the equation (1): the two-dimensional Toda lattice. A connection with Nevanlinna theory was established and explored, which permitted to generalize the results of Jost and Wang [31].

2. Exceptional values in holomorphic families of entire functions, math.CV/0503750

This contains answers to questions of G. Julia (1926), and A. Sokal about the dependence of Picard's exceptional values on parameter in holomorphic families of entire functions.

3. Meromorphic traveling wave solutions of the Kuramoto-Sivashinsky equation, nlin.SI/0504053

It is shown that all these solutions have to be elliptic functions, possibly degenerate. This permits, at least in principle, to find all these solutions explicitly. It turns out that there are no other meromorphic solutions except those explicit solutions found earlier by physicists. The method to prove this was invented by the proposer in 1984 and applied to another differential equation, but now it become clear that it applies to many interesting differential equations. The full range of applicability of the argument still has to be investigated.

Human resources development . Two of the proposer's PhD students graduated in the last 5 years: Sergei Merenkov in 2003 and Byung-Geun Oh in 2004. Both are continuing active research on postdoctoral positions. Two advanced courses "Topics in Geometric Function Theory" were taught by the proposer in Purdue University (last time in spring 2005). These courses were partially based on the proposer's research funded by NSF. A mini-course "Introduction to holomorphic dynamics" was taught to Russian and Israeli graduate and undergraduate students in spring 2004 when the proposer visited Weizmann Institute (Rehovot, Israel) during his sabbatical leave. This mini-course was also partially based on the proposer's research funded by NSF.

PROJECT DESCRIPTION
Real meromorphic functions

1. The B. and M. Shapiro Conjecture and related questions.

Problems in sections 1 and 2 are closely related to several areas of mathematics (Analysis, Algebraic geometry, Combinatorics and Control theory). The PI plans to continue his work on these problems in close collaboration with A. Gabrielov (Purdue Universirty). In the present proposal, the PI emphasizes analytic aspects of the topic. Combinatorial and topological aspects are reflected in a separate proposal of A. Gabrielov, submitted to the Algebra, Number theory and Combinatorics Program of DMS.

A meromorphic function is called *real* if it maps the real line into itself.

Conjecture 1. *Let $f = (f_1, \ldots, f_p)$ be a vector of complex polynomials in one variable, and*

$$W = W(f) = \begin{vmatrix} f_1 & \ldots & f_p \\ f_1' & \ldots & f_p' \\ & \ldots & \\ f_1^{(p-1)} & \ldots & f_p^{(p-1)} \end{vmatrix}$$

their Wronski determinant. If all zeros of W are real then there exists a matrix $A \in GL(p, \mathbf{C})$ such that pA is a vector of real polynomials.

In a more geometric language, *if all inflection points of a rational curve $f : \mathbf{P}^1 \to \mathbf{P}^{p-1}$ are real then f can be made real by an automorphism of* $\mathbf{P}^{p-1}$. When $p = 2$ this means: *if all critical points of a rational function f are real then there exists a fractional-linear transformation ϕ such that $\phi \circ f$ is real.*

This is one of the central conjectures in real enumerative geometry; the surveys of this subject are [61, 62], and it has applications to real Schubert Calculus, geometry of real plane algebraic curves and to linear control theory.

In [14] we proved the conjecture for $p = 2$. The proof is quite complicated; it uses arguments from several different areas, but its main drawback is the use of the Uniformisation theorem, which was the main obstacle to generalizations to higher dimensions. The attempts to find another proof of this theorem occupied 5 years (the paper [13] and the preprint 1 above can be considered as by-products of these attempts, see Section 5). In August 2005 we finally found a new proof of this result, which does not use the

Uniformization theorem. We believe that this new proof opens the way to the multidimensional case, at least to the case $p = 3$, which is needed for applications to real plane algebraic curves [33].

Vectors f of p linearly independent polynomials of degree at most $m + p - 1$, modulo the equivalence relation $f \sim g$ if $f = gA,\ A \in GL(p, \mathbf{C})$ represent points of the Grassmannian $G = G(p, m + p)$. The elementary properties of the Wronski determinant imply that the polynomial $W(f)$ has degree mp, and multiplication of f by a constant non-singular matrix A results in multiplication of $W(f)$ by the non-zero constant $\det A$. So the map $f \mapsto W(p)$ induces a well defined map

$$W : G \to \mathbf{P}^{mp}, \tag{2}$$

which we call the *Wronski map*. This is a finite regular map between compact complex analytic manifolds. Wronski map also arises in mathematical physics [45, 57, 56]. Its degree (the number of preimages of a generic point) will be called the *complex degree* of the Wronski map. It can be computed using the Schubert calculus. If the Grassmannian is embedded to the appropriate projective space via the Plücker embedding as the *Gramssmann variety* [24], the Wronski map becomes a restriction of a linear projection on this Grassmann variety.

The following problem goes back to [23] where it was stated for $p = 2$:

Question 1. *How to characterize the ramification locus of the map W, or the image of this locus under W?*

A reasonable answer to this question should imply Conjecture 1, because as we showed in [14, 15, 16], Conjecture 1 is equivalent to the following: the image of the ramification locus does not intersect the set of polynomials whose all zeros are real. At the time of this writing, there is no plausible conjectures about the answer to Question 1, even for $p = 2$. We propose to begin the investigation with the following:

Question 2. *What is the degree of the ramification locus?*

We intend to use quasi-homogeneity methods and asymptotic analysis of the type performed in [60, 17], and expect that Question 2 can be answered.

Restriction of the Wronski map to the real part of G is called the *real Wronski map* and it maps the real Grassmannian to the real projective space. Conjecture 1 means that the preimage of any polynomial with all zeros real under the real Wronski map has maximal cardinality equal to the complex degree of W. In [15, 16] we found the *real degree* of the Wronski map, which is essentially the topological (Brouwer) degree, properly modified to take into account the non-orientability of the domain and the range. This real degree turns out to be non-zero when $m + p$ is odd, so in this case we have a lower estimate of the number of preimages of a point under the real Wronski map. According to [60, 30], this was the first instance that a non-trivial lower estimate in a problem of real enumerative geometry was obtained. On the other hand, in [18] we showed that the Real Wronski map is not surjective when both m and p are even. So the following question remains:

Question 3. *Is the real Wronski map surjective hen both m and p are odd?*

The simplest unknown case is $(m, p) = (3, 3)$. The answer will have implications for all three areas where the real Wronski map plays a role: control theory (see below), plane real rational curves [33] and real Schubert calculus.

2. Control of linear systems by static output feedback.

The problems considered in this section have origin in the control theory. However the proposer works on them because of their intrinsic mathematical interest, as problems of Analysis, the possible significance of these problems for engineering was not the main criterion for their selection and it is not discussed here.

Linear systems we consider here are of the simplest kind: they are described by equations

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{3}$$

where x, u and y are functions of a real variable (time) with values in $\mathbf{R}^n, \mathbf{R}^m$ and $\mathbf{R}^p$, respectively, and A, B, C are constant real matrices of appropriate sizes. The functions x, u, y are interpreted as inner state, input and output, respectively. *Static output feedback* means adding the equation

$$u = Ky \tag{4}$$

where K is a constant $m \times p$ matrix, called the *compensator*. The matrices A, B, C are considered as given, and the designer chooses K. The goal of adding the feedback is to achieve the desired spectral properties of the system. After elimination of u and y from (3), (4) one obtains the equation $\dot{x} = (A + BKC)x$ whose characteristic polynomial is $\phi_K(\lambda) = \det(\lambda I - A - BKC)$. The most ambitious goal of the designer is solving the *pole placement problem*: to find K such that the roots of ϕ_K (=eigenvalues of the system) occupy prescribed positions in the complex plane. So the pole placement problem is solvable if the map $K \mapsto \phi_K$ is surjective. Here one can consider real or complex K, and the problem with complex K was completely solved in 1980-s, see, for example [9]. But engineering applications in many cases impose the restriction that the matrix K should be real. Dimension count shows that the pole placement problem is in general unsolvable if $n > mp$. That it is solvable for $n < mp$ (with real K) is a result of A. Wang [65]. So only the so-called *critical case* $n = mp$ remains, and we restrict ourselves to this case.

After a natural factorization, the *pole placement map* $K \mapsto \phi_K$ becomes a map of the type (2). Moreover, if the Grassmannian is realized by the Plücker embedding, the pole placement map becomes a restriction of a linear projection on the Grassmann variety. This projection depends on the given system (A, B, C). Not all projections arise in this way, so we have the following

Question 4. *Which projections of the Grassmannian arise as pole placement maps of linear systems?*

The Wronski map turns out to be one of such projections, which permitted us to obtain negative results in control theory in [17, 18]. More precisely, we proved that the pole placement map of a generic system (A, B, C) with $n = mp$ and both m and p even, omits an open set of real polynomials. (Previously this was known only for $(m, p) = (2, 2)$ and $(4, 2)$ [53], and there was even a conjecture that these are the only exceptional cases). Our negative results depend on the known cases of Conjecture 1. A progress in this conjecture is likely to lead to new negative results on the pole placement problem.

To obtain positive results, we need some progress in Question 4. Then we hope to apply the machinery of the real degree computations [15, 16] to compute the degrees of the real Grassmann variety projections, other than the Wronski map.

From the point of view of applications, the following problem is even more important than pole placement:

Stabilizability problem. *For a given system (A, B, C), is it possible to choose K so that all roots of p_K lie in the left half-plane?*

Very little is known about this, except for the case that $\min\{m, p\} = 1$ when the pole placement map is linear [9]. It is known that there is an open set of non-stabilizable systems with $(m, p) = (2, 2)$ [66]. Pole placement is a stronger requirement than stabilizability, so a positive result on the pole placement implies a positive result on stabilizability. We hope to be able to produce negative results on stabilizability by using some modification of the Wronski map in the same way as the Wronski map was used in [17, 18]. (The Wronski map itself does not give such counter-examples).

As the last problem in this section, we mention the *dynamic output feedback*. This means that the equation (4) is replaced by a linear differential equation with constant coefficients. A parallel theory exists for this case, with complex output feedback, where the ordinary Grassmannian is replaced by a "quantum Grassmannian", which is a Grassmannian over a polynomial ring [54]. There is a natural analog of the Wronski map in this setting, and our results on the real degree from [15, 16] extend to this case (work in progress).

3. Zeros of derivatives of real entire functions.

We recall that the Laguerre–Pólya class LP consists of real entire functions which can be approximated uniformly on compact subsets of the plane by polynomials whose zeros are real. Evidently the class LP is closed under the differentiation. Pólya obtained a parametric description of this class: it consists of canonical products of genus 1 with real zeros, multiplied by $\exp(-az^2 + bz + c)$ where $a \leq 0$ and b, c are real. Thus all functions of the class LP have at most order 2, normal type. Laguerre–Pólya class is indispensable in many questions of analysis, including total positivity, harmonic analysis and spectral theory of differential operators [32, 43, 29].

The investigation of location of zeros of successive derivatives of real entire functions begins with the two influential papers of Pólya [50, 51]. Problems proposed by Pólya inspired a lot of research in XX century, and only recently some of them were completely solved [58, 10, 34, 6]. In 2005, M. Berry, a physicist, published an interesting paper [3] where he gives a new interpretation of the results of Pólya, in terms of infinite dimensional dynamical systems.

In [50] Pólya writes about successive differentiation of real entire functions: *"The real axis seems to exert an influence on the complex zeros of $f^{(n)}$; it seems to attract these zeros when the order is less than* 2*, and it seems to repel them then the order is greater than* 2. A very precise version of the first part, known as the Pólya–Wiman conjecture, was recently proved by Kim and Ki [34]. For the second part Pólya made the following precise

Conjecture 2. *If the order of the real entire function is greater than 2, and f has only a finite number of non-real zeros, then the number of non-real zeros of $f^{(n)}$ tends to infinity as $n \to \infty$.*

This conjecture is still open. Pólya himself did the case when f is of finite genus and the total number of zeros of f is finite [50]; this was extended by McLeod [42] to the case when the genus of the canonical product in f is less than the genus of f by two. Further generalization of this result is in [22]. Pólya used the saddle-point asymptotic method, and Gethner and McLeod a very refined version of this method due to Hayman [25]. In both cases, the authors prove much more then required: they find the so-called "final set" which is the limit set of zeros of successive derivatives.

Some cruder but more general method is required to establish the conjecture. Inspired by [3], we hope this can be done by combining arguments from [6] with the potential-theoretic approach described in [12].

The main result of [6] is the following: *Let f be a real entire function of infinite order, with only finitely many non-real zeros. Then f'' has infinitely many non-real zeros.* Combined with the result of Sheil-Small [58] for functions of finite order, this establishes a conjecture of Wiman (1911): *if f is a real entire function such that ff'' has only real zeros, then f belongs to LP.* A major previous achievement was the result of Hellerstein and Williamson [27, 28] that a real entire function such that $ff'f''$ has only real zeros, belongs to LP.

Recently J. Langley [37] was able to generalize our theorem by replacing in it the second derivative by any derivative. This implies that Conjecture 2 is true for functions of infinite order. The strongest result of this type for functions of finite order belongs to Edwards and Hellerstein: for real entire functions of finite order with finitely many real zeros, they produce a positive lower estimate for the number of non-real zeros of $f^{(n)}$, but unfortunately their estimate does not tend to infinity as $n \to \infty$, while f is fixed. It seems hard or impossible to prove the desired lower estimate for each n, but we hope to be able to prove the asymptotic result, that the number of non-real zeros tends to infinity as $n \to \infty$.

4. Oscillation of real functions and their spectra

The problems considered here go back to Sturm, with important contributions made in XX century by Pólya and others. Recently there was a surge of interest due to connections with singularity theory [1] and projective differential topology [49].

For a function in $L^1(\mathbf{R})$, its spectrum can be defined as the support of its Fourier transform. For locally integrable functions of sub-exponential growth,

$$\int_{\mathbf{R}} |f(t)|e^{-\epsilon|t|}dt < \infty, \tag{5}$$

one can follow Carleman's approach: the functions $F^+(z) = \int_{-\infty}^0 f(x)e^{-itz}dt$ and $F^-(z) = -\int_0^\infty f(t)e^{-izt}dt$ are analytic in the upper and lower half-planes, respectively. If the ordinary Fourier transform of f exists, it is equal to the difference of boundary values of F^+ and F^-, so we define the *spectrum* of f as the smallest closed set $S \subset \mathbf{R}$ such that F^+ admits an analytic continuation to F^- through $\mathbf{R}\backslash S$. This is consistent with all other definitions of a spectrum using various generalizations of Fourier transform. A *spectral gap* is a symmetric interval disjoint from the spectrum.

There is a general principle that a real function f having a *spectral gap* $(-a, a)$ has to oscillate with the frequency at least a/π. For periodic functions this is a classical theorem of Sturm and Hurwitz. First generalizations of this to non-periodic functions are due to M. Krein and B. Levin [38] and B. Logan [40]. B. Logan stated the following simple conjecture: *If f is an integrable function with a spectral gap $(-a, a)$, then the lower density of its sign changes is at least a/π.* He himself proved this under the additional condition that the spectrum of f is bounded; this is a very restrictive condition which implies that f is an entire function of exponential type. Since 1965 there was no results on the Logan's conjecture with lower density, though the problem was repeated in several places, for example, in [2]. The only known results of this type (Krein, Levin, Ostrovskii and Ulanovskii) were using some integrated densities which are greater than or equal to the lower density [48].

In the paper [20], we confirmed Logan's conjecture, moreover, we found the exact growth restriction under which it holds. It turns out to be a slightly stronger restriction than (5), namely f should be integrable against a *non-quasianalytic weight* $e^{-\omega}$, where $\omega > 0$ is a sufficiently regular (for example, even and increasing as a function of $|x|$) function with the property

$$\int_{\mathbf{R}} \omega(t)(1+t^2)^{-1}dt < \infty.$$

For functions of faster growth (but still satisfying (5)), Logan's conjecture fails, even with the additional requirement of boundedness of spectrum.

There are several directions of developing these results. First of all, *some* estimate from below of the lower density of sign changes still holds for all functions with a spectral gap satisfying (5). We propose to begin with functions with bounded spectra (=entire functions of exponential type).

Question 5. *What is the best possible lower estimate for the lower density of sign changes of a real entire function of exponential type with a spectral gap $(-a, a)$?*

This is equivalent to the following question from the theory of entire functions: *Let f be an entire function of exponential type with indicator diagram $[-ia, ia]$. What is the precise upper estimate for the upper density of zeros of such function?* It looks like a very basic extremal problem of the general theory of entire functions (see, for example, [38]) but the answer is unknown. One possible approach to it is based on the use of the Hilbert transform as in [20].

The lower density is $\liminf s(r, f)/r$, where $s(r, f)$ is the number of sign changes on the interval $(0, r)$. For discontinuous functions $s(r, f)$ is defined as the minimal degree of polynomials p such that the restriction of pf on $(0, r)$ is non-negative in the sense of distributions. Lower density is the simplest one, but there are many other types of densities that naturally arise in such questions. Experience shows that for each asymptotic problem with a sequence of real numbers there is its own adequate density, see [36]. There are some indications that the appropriate density for the Logan's problem could be the so-called Lower Beurling–Malliavin density [46, 48], which does not exceed the lower density.

The third, and probably most promising direction is relaxing the condition that the function has a spectral gap. Such possibility was demonstrated in the work [46, 47] where the authors only assume that the Fourier transform $\hat{f}$ of f is real analytic on $(-a, a)$. The conclusion is that f oscillates much *or* that $\hat{f}$ is analytic on the whole real line. This result does not contain Logan's conjecture because the rate of oscillation was measured with (upper) Beurling–Malliavin density, which can be much larger than the lower density. We expect that this result should hold with some kind of lower density. The methods of [46, 47] and [20] are very different, so new ideas are required to obtain such result.

Recalling the Carleman definition of the spectral gap above, we conclude that the last problem is related to the following statement: if a real function on $[0, \infty)$, satisfying (5) has few changes of sign in some sense and its Laplace transform has an analytic continuation to a neighborhood of a sufficiently long (depending on the density of the sign changes) symmetric interval around zero on the imaginary axis, then it has an analytic continuation to the neighborhood of the whole imaginary axis. This can be considered as a generalization of the classical Marcinkiewicz theorem (see, for example, [39]) from probability theory which deals with functions having no changes of sign at all. The discrete version, with a power series instead of the Laplace transform, is the famous Fabry theorem [4]. (The better-known part of this Fabry theorem deals with gaps, but the original version contains also a sign change condition). No satisfactory generalization of Fabry theorem to Laplace transform was obtained so far, though there were attempts in this direction [21].

Finally we mention the hardest problem in this section: what is the proper multi-dimensional generalization of the Sturm theorem? Which known properties of the nodal domains of eigenfunctions with high eigenvalues are shared by functions with a spectral gap? To begin with, we propose the following

Conjecture 3. *Let $f(x,y) = \sum c_{m,n} exp(i(mx+ny))$ be a real function on the unit torus with a spectral gap, which means that $a_{m,n} = 0$ for $m^2+n^2 < r^2$ for some large r. Does this imply that the set where $f(x,y) > 0$ cannot contain a disc of radius R, where R depends on r and $R \to 0$ as $r \to \infty$?*

5. Metrics of positive curvature on the sphere with conic singularities.

A Riemannian metric on a surface has a conic singularity at a point if in some conformal coordinate z at this point the metric is given by $ds^2 = g(z)z^{2\alpha-2}$, where g is smooth and positive. The neighborhood of this singularity looks like a cone with the total angle around the singular point equal to $2\pi\alpha$. The question we are going to discuss is

Question 6. *Under what conditions there exists a metric of constant curvature $+1$, in the same conformal class as the standard spherical metric, with conic singularities of prescribed angles $2\pi\alpha_j$ at prescribed points z_j*

The similar problem with zero and negative curvature is completely solved for arbitrary compact surfaces. The problem was treated for the first time in the late XIX century by Picard in connection with the Uniformization theorem, but a complete rigorous solution was achieved only recently in the paper of Troyanov [64]. Actually Troyanov deals with the more general problem of prescribing a variable curvature. It turns out that there is only one obstacle in the case of non-positive curvature: the Gauss–Bonnet theorem. If the necessary condition coming from the Gauss–Bonnet theorem is satisfied, a solution exists and is unique. In the case of positive curvature on the sphere, Troyanov's results apply when the angles at all singularities are small enough (less than 2π).

Question 6 is completely solved for the number of singularities at most 2 in [63] and 3 in [13]. This case was treated by F. Klein [35] but his results are incomplete. Later the question was investigated by many authors, including physicists, with various additional conditions, but the proposer could not find the complete result stated in the literature before [13]. What makes the case of three points easier is of course that there is only one configuration of three points on the sphere, up to conformal equivalence.

Another known case is that all angles at the singularities are multiples of 2π. One can show that all such metrics have the form $|f'|^2/(1+|f|^2)^2|dz|^2$, where f is a rational function. Singularities of the metric are exactly the critical points of f, and much is known about sets of rational functions with prescribed critical points is know [23, 56, 19]. Exploring this connection with critical points of rational functions, and using the results of [14, 17] the proposer gave first examples of non-uniqueness and "break of symmetry" in Question 6.

A special case of Question 6 is when all prescribed singularities lie on a circle. By conformal invariance we may assume that this circle is the real line. Then the upper half-plane with the metric in question becomes a spherical polygon, and Question 6 in this case is equivalent to the question of existence of a spherical polygon of prescribed conformal type with prescribed angles. (For flat polygons, the existence and uniqueness follows from the Schwartz–Christoffel formula, and for hyperbolic polygons, from Troyanov's theorem). A complete description of rational functions with prescribed real critical points is known now [19] which answers Question 6 for the case of real points z_j and integer α_j.

We hope that the approach based on analytic theory of differential equations, as in [13], combined with some "continuity method" as in [14] will permit to solve Question 6. The original motivation for the problem was an attempt to find a new proof of Conjecture 1 for $p = 2$, but the Question 6 seems to be interesting by itself, and the proposer intends to investigate it.

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