PROJECT SUMMARY

Alexandre Eremenko and Andrei Gabrielov

Meromorphic functions and their applications

Submitting Organization: Purdue University

The proposers intend to continue their study of the distribution of roots and critical points of meromorphic functions, using the methods of geometric theory of meromorphic functions and topology, the approach which already brought significant results in their previous research. All problems in this proposal arise naturally from the results of the previous research of the proposers funded by NSF. In this proposal the term "meromorphic function" denotes a function meromorphic in the complex plane, that is a ratio of two entire functions.

The proposers plan to concentrate on the following specific problems:

Geometry and topology of real and complex spectral loci for families of one-dimensional Schrödinger operators with polynomial or rational potentials.

Singularities of implicit analytic functions defined by entire relations.

General study of certain classes of meromorphic functions that occur in holomorphic dynamics and in spectral theory of Schrödinger operators with polynomial potentials.

Asymptotic distribution of zeros of sequences of entire functions.

Uniform polynomial approximation of discontinuous functions on systems of intervals.

Dynamics of rational functions, especially analytic invariant curves of such functions.

Proposed research will expand our understanding of solutions of transcendental equations that occur in analysis and mathematical physics.

The broader impact of the theory of meromorphic functions consists in their use in many areas of applied mathematics from mathematical physics to computer science. The choice of the problems in this proposal is motivated by their pure mathematical interest. However the previous work of the authors on this subject already found applications, sometimes unexpected, in control theory, material science, computer science, signal processing, mathematical physics and astrophysics.

This proposal contains several problems of function theory that arise in mathematical physics, namely in quantum mechanics, and the proposers expect their results to be used by physicists.

RESULTS FROM PRIOR NSF SUPPORT

A. Eremenko was supported by the NSF grant DMS–0555279 (funding period: June 1, 2006 – May 31, 2011). The research was titled "Real Meromorphic Functions"; the amount of support was $335,023. The results of this research are contained in the 22 papers listed below (18 published and 4 accepted for publication). In addition, two preprints are posted on the arxiv, but not submitted to journals yet.

A. Gabrielov was supported by the NSF grant DMS-0801050 (funding period: June 1, 2008 – May 31, 2011). The research was titled "Homotopy, Complexity and O-Minimality"; the amount of support was $219,000. The results of this research are contained in 6 papers and 5 preprints. Below we only list his joint papers with A. Eremenko and M. Azar, which are relevant for this proposal.

1. A. Bergweiler and A. Eremenko, Proof of a conjecture of Pólya on the zeros of successive derivatives of real entire functions, *Acta math.*, 197, 2 (2006), 125-146.

This is the final step which completes the proof of a conjecture of Pólya of 1943. Combined with the previous results of Langley [52] and Kim [50], this result implies a remarkable alternative: *Let f be an arbitrary real entire function, and let $N(f^n)$ denote the number of non-real zeros of the n-th derivative. Then either $N(f^n) = 0$ for all sufficiently large n, or $N(f^n) \to \infty$ as $n \to \infty$.* The first statement holds iff $f = pg$ where p is a real polynomial and g is in the Laguerre–Pólya class, that is g is a limit of polynomials with all zeros real.

2. A. Eremenko, A Markov-type inequality for arbitrary plane continua, *Proc. AMS*, 135 (2007) 1505-1510.

Classical inequality of Markov on polynomials on an interval is extended by replacing the interval by an arbitrary continuum in the complex plane.

3. A. Eremenko and P. Yuditskii, Uniform approximation of sgn(x) by polynomials and entire functions, *J. d'Analyse Math.*, 101 (2007) 313-324.

We found a precise asymptotics as $d \to \infty$ of the error of the best uniform approximation of the function $\mathrm{sgn}(x)$ by polynomials of degree d on the set $[-1, -a] \cup [a, 1]$. The result already found applications in computer science [65, 25] and was generalized by Nazarov, Volberg and Yuditskii [60].

4. A. Eremenko, N. Nadirashvili and M. Yakobson, On nodal sets and nodal domains in the sphere and in the plane, *Ann. Inst. Fourier*, 57 (2007) 2345-2360.

All possible topological types of the nodal sets of spherical harmonics are determined.

5. A. Eremenko and I. Ostrovskii, On the “pits effect” of Littlewood and Offord, *Bull. London Math. Soc.*, 39 (2007) 929-939.

6. A. Eremenko, A. Gabrielov and B. Shapiro, High energy eigenfunctions of one-dimensional Schrodinger operators with polynomial coefficients, *Computational Methods and Function Theory*, 8 (2008). No. 2, 513–529.

The limit distribution of complex zeros of eigenfunctions of polynomial Hermitian and PT-symmetric oscillators as the eigenvalue tends to infinity is found. This distribution depends only on the top term the polynomial potential and on the boundary conditions.

7. A. Eremenko, A. Gabrielov and B. Shapiro, Zeros of eigenfunctions of some anharmonic oscillators, *Ann. Inst. Fourier, Grenoble,* 58, 2 (2008) 603-624.

We prove that all complex zeros of eigenfunctions of the even quartic oscillator

$$-y'' + (z^4 + az^2)y = \lambda y, \quad y(\pm\infty) = 0 \tag{1}$$

with real a, belong to the union of the real and imaginary axes. The proof uses a new method based on a topological characterization of eigenfunctions.

8. A. Eremenko and A. Gabrielov, Analytic continuation of eigenvalues of a quartic oscillator, *Comm. Math. Phys.*, v. 287, No. 2 (2009) 431-457.

The method of paper 7 is applied to give a rigorous proof of the facts discovered by physicists in 1969 [11]: The spectral locus of the even quartic family (1) (with complex a) consists of exactly two connected components. These components are smooth, and all singularities of the eigenvalue λ, as a function of the complex parameter a, are algebraic branch points which accumulate only to infinity (See the Project description on this).

9. A. Eremenko, P. Yuditskii, Extremal problem for a class of entire functions, *CR Acad. Sci. Ser. I*, 346 (2008) 825-828.

The maximum value of the density of zeros of entire functions of exponential type with a given indicator diagram $[-i\sigma, i\sigma]$ is found. This class of entire functions occurs in harmonic analysis: they are Fourier transforms of hyperfunctions with support on $[-\sigma, \sigma]$. The paper answers a question asked in [38]

10. A. Eremenko, Fabry's theorem for power series with regularly varying coefficients, *Proc. AMS*, 136 (2008), 4389-4394.
11. A. Eremenko, Densities in Fabry's theorem, *Illinois J. Math.*, 52, No. 4, 1277-1290 (2008).

These two papers contain a new approach to the classical theorem of Fabry on singularities of a power series on its circle of convergence. Precise conditions on the sign changes of coefficients of a power series with radius of convergence 1 which ensure the existence of a singularity on a given arc of the unit circle are studied. These conditions improve the results of Fabry.

12. W. Bergweiler and A. Eremenko, Direct singularities and completely invariant domains of entire functions, *Illinois J. Math.* 52 (2008) N 1, 243-259.

Direct singularities of inverses of entire functions and their relation to dynamics of these entire functions are studied.

13. A. Eremenko, L. Liao and T. Ng, Meromorphic solutions of higher order Briot-Bouquet differential equations, *Math. Proc. Cambridge Philos. Soc.*, v. 146, no. 1 (2009) 197-206.

All meromorphic functions having at least one pole and satisfying a differential equation of the form $F(y^{(k)}, y) = 0$, where F is a polynomial, are explicitly described. It turns out that all these meromorphic functions are elliptic, possibly degenerate. This description was conjectured by E. Hille and A. Eremenko in 1970-s. The method developed for this problem found several applications in the theory of integrable systems, see [19, 28, 51, 72], where equations relevant to physics are treated.

14. W. Bergweiler and A. Eremenko, Meromorphic functions with linearly distributed values and Julia sets of rational functions, *Proc. AMS.* 137 (2009), 2329-2333.

15. A. Eremenko and S. van Strien, Rational functions with real multipliers, accepted in *Trans. AMS*

These two papers are closely related. In the first paper, we proved that if a relatively open part of the Julia set of a rational function belongs to a smooth curve then the whole Julia set must be contained in a circle. This generalizes one of the main results of Fatou [42] on dynamics of rational functions. The proof explores the connection between a rational function and its Poincaré function. Another proof is given in the second paper.

In the second paper, we also use Poincaré functions to prove that whenever the multipliers of repelling periodic points are all real, the Julia set must lie in a circle. Preliminary classification of such functions is made.

16. W. Bergweiler and A. Eremenko, On the number of solutions of a transcendental equation arising in the theory of gravitational lensing, *Comput. Methods and Funct. Theory* 10 (2010), No. 1, 303–324.

Using the theory of harmonic maps, we show that certain transcendental equation occurring in the theory of gravitational lensing has at most 6 solutions, and any number of solutions between 1 and 6 can actually occur.

17. A. Eremenko and A. Gabrielov, Tangencies between holomorphic maps and holomorphic laminations, *Proc. AMS* 138 (2010), 2489-2492.

This solves a problem arising in holomorphic dynamics: *the set where a holomorphic map intersects a holomorphic lamination non-transversally is analytic.* This was proved in [3] for holomorphic curves; we extend it to maps of any dimension.

18. W. Cherry and A. Eremenko, Landau's theorem for holomorphic curves in projective space and the Kobayashi metric on hyperplane complement, *Pure and Appl. Math. Quarterly.*, 7 (2011) 199–221.

Using a potential-theoretic approach developed by A. Eremenko and M. Sodin in 1980-s, we give an explicit estimate of the Kobayashi metric of the complement of $2n+1$ hyperplanes in projective space of dimension n. Hyperbolicity of this complement was known long ago, but all previous proofs were non-constructive and gave no explicit estimates. This solves a problem of Dufresnoy (1944).

19. A. Eremenko, Brody curves omitting hyperplanes, accepted in *Ann. Acad. Sci. Fenn., Math.*

Using the same potential-theoretic method as in paper 18, the following result is proved: *A holomorphic curve in* $\mathbf{P}^n$ *with bounded derivative with respect to the Fubini–Study metric, omitting* n *hyperplanes in general position, has growth at most of order one, normal type.* The number of omitted hyperplanes here is optimal. This result was previously known only for $n = 1$ [18], or when $n+1$ hyperplanes are omitted [16].

20. W. Bergweiler and A. Eremenko, Dynamics of a higher dimensional analog of trigonometric functions, accepted in *Ann. Acad. Sci. Fenn. Math.*

The so-called "dimension paradox" in the dynamics of entire functions is extended to a class of quasiregular mappings in $\mathbf{R}^n$, $n \geq 3$. For every $n \geq 2$ there exists a dynamically defined partition of $\mathbf{R}^n$ into curves (properly embedded rays $[0, \infty)$) such that the union of the curves without their endpoints has Hausdorff dimension 1.

21. A. Eremenko and J. Langley, A survey of some results after 1970. Appendix to the book: A. Goldberg and I. Ostrovski, *Distribution of values of meromorphic functions*, Transl. Math. Monogr., vol. 236, AMS, Providence, RI, 2008.

A survey of the theory of meromorphic functions in the last 40 years.

22. A. Eremenko and A. Gabrielov, Elementary proof of the B. and M. Shapiro conjecture for rational functions, in the book: *Notions of positivity and the geometry of polynomials* to be published by Birkhauser.

A new, simplified and elementary proof of our result [31] is given. This new proof may be suitable for further generalizations of the B. and M. Shapiro conjecture, see [36, 69], and the next preprint:

23. M. Azar and A. Gabrielov, Some lower bounds in the B. and M. Shapiro conjecture for flag varieties, arXiv:1006.0664.

This is a further development of the previous work of A. Eremenko and A. Gabrielov [32, 36] on the lower estimates in real Schubert calculus.

24. A. Eremenko and A. Gabrielov, Irreducibility of some spectral determinants, arXiv:0904.1714.

25. A. Eremenko and A. Gabrielov, Singular perturbation of polynomial potentials in the complex domain with applications to PT-symmetric families, arXiv:1005.1696

These two preprints will be discussed in the Project description.

Human resources development. Currently A. Eremenko has two PhD students, both started in 2009. Matthew Barrett is working on holomorphic curves in projective space. The purpose is to obtain a complete generalization of the result of Clunie and Hayman [18] from dimension 1 to arbitrary dimension: *for a holomorphic curve in* $\mathbf{P}^n$ *omitting* n *hyperplanes in general position, condition* $\|f'\|(z) = O(|z|^\sigma)$ *implies* $T(r, f) = O(r^{\sigma+1})$. A partial result was obtained in the paper 19 above. The proof in 19 has two drawbacks: it only deals with the case of bounded Fubini–Study derivative, and it is non-constructive. There drawbacks will be removed in a forthcoming joint paper. Matthew is expected to defend his thesis by 2012.

Koushik Ramachandran is working on Martin functions in unbounded domains in $\mathbf{R}^n$. A Martin function is a positive harmonic function, zero on the boundary. It is important to estimate the growth of a Martin function which depends on the shape of the domain. A very general estimate from below can be obtained by a method of Carleman which works in any dimension. However the usual method for an upper estimate, based on a theorem of S. Warschawski, only works in dimension 2. The student is expected to produce upper estimates in arbitrary dimension, at least for "nice" regions which are close to cylinders or cones in certain sense. This problem arises in probability theory [4].

A. Gabrielov advised a graduate student, Monique Azar. She defended her thesis in 2008 and then stayed at Purdue until 2009 as a postdoc. A preprint based on her thesis has been posted in 2010 (item 23 above). In spring 2010, A. Gabrielov advised an exchange student from Sweden, Per Alexandersson. Their joint paper is in preparation; it will be discussed in the Project description.

Both proposers regularly teach advanced graduate courses partially based on their research funded by NSF.

PROJECT DESCRIPTION
Meromorphic functions and their applications

1. Eigenvalue problems in the complex plane.

We consider eigenvalue problems of the form

$$-y'' + P(z,a)y = \lambda y, \quad y(z) \to 0,\ z \to \infty, \quad z \in L_1 \cup L_2. \tag{2}$$

Here P is a polynomial in z depending analytically on complex parameter a, and L_j are two rays in the complex plane. When the coefficients of P are real and $L_1 \cup L_2 = \mathbf{R}$, the problem is Hermitian. Under certain conditions on the top degree coefficient of P and on the rays L_j, the problem has an infinite discrete sequence of simple eigenvalues λ_j. Problem (2) was considered in full generality for the first time by Sibuya [67]. Relevance of this problem to physics became clear from [11, 68, 10]. If P is a monic polynomial, then there exists an entire function $F(a, \lambda)$, which is called the *spectral determinant*, such that the eigenvalues are given by the equation

$$F(a, \lambda) = 0. \tag{3}$$

The set of (a, λ) defined by this equation is called the *spectral locus*. The general problem described in this section is the study of geometry and topology of such spectral loci for various families of potentials $P(z, a)$. The rays L_1, L_2 are called *admissible* if $F \neq 0$. There is a simple description of all pairs of admissible rays for any given P. For admissible rays, the spectral locus is non-empty.

For a family of even quartic oscillators (1), we proved in [33] that the spectral locus consists of exactly two smooth hypersurfaces, the fact discovered in 1969 by Bender and Wu [11] by heuristic arguments combined with computation. This was extended to several other one-parametric families of cubic, quartic and sextic oscillators in [34]. One problem is to extend the result to multi-parametric families:

Conjecture 1. *Let $P(z, a),\ a \in \mathbf{C}^{d-1}$ be the family of all monic potentials of degree d, and L_1, L_2 are admissible. Then the corresponding spectral locus is connected.*

Conjecture 2. *Let $P(z, a),\ a \in \mathbf{C}^{d-1}$ be the family of all even monic potentials of degree $2d$, and L_1, L_2 are admissible. Then the spectral locus consists of two components.*

The proof of both conjectures is almost complete by now. It will be contained in a paper by P. Alexandersson and A. Gabrielov (in preparation).

The methods that we use are completely different from the perturbative methods of all previous work on these problems [22, 23, 24]. Our method is global and non-perturbative. It uses what we call the "Nevanlinna parametrization", based on R. Nevanlinna's work [61]. If y is an eigenfunction of (2), and y_1 is another linearly independent solution, then $f = y/y_1$ is a meromorphic function which has no critical points and exactly $\deg P + 2$ *asymptotic tracts.* Asymptotic tracts are sectors in the z-plane in which $f(z)$ has a limit as $z \to \infty$. These limits are called *asymptotic values.* Due to the boundary conditions, two of these tracts that contain the rays L_1, L_2 have zero asymptotic value. We conclude that

$$f : \mathbf{C}\backslash f^{-1}(A) \to \bar{\mathbf{C}}\backslash A, \tag{4}$$

where A is the finite set of asymptotic values, is a covering map. The main result of Nevanlinna says that such map f is essentially determined by its topological properties and by the set A. Moreover, for every local homeomorphism g like (4) with finitely many tracts, there exists a meromorphic function $f = g \circ \phi$, where ϕ is a homeomorphism of $\mathbf{C}$. Up to a normalization, this meromorphic function is determined by g and A uniquely. The parameters (a, λ) of (2) can be recovered from the function f by the formula

$$\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(P(.,a) - \lambda).$$

This permits one to construct a *parametrization* of the spectral locus by a set G of meromorphic functions f without critical points and with $\deg P + 2$ asymptotic tracts. Topology of the set G can be studied by combinatorial methods: coverings of the form (4) are encoded by certain graphs embedded in the plane.

In [33], we find the monodromy of the map Φ which assigns to a function $f \in G$ its asymptotic values. The monodromy action has two transitivity classes which implies that G (and thus the spectral locus) consist of two components. However the monodromy action that we computed in [33] for the even quartic potential contains much more information on the spectral locus, and to recover this information is our next problem. It is likely that each component of G is a surface of infinite genus with infinitely many ends. The results in [33] clearly show that it must consist of periodic and doubly periodic pieces.

Conjecture 3. *The spectral locus of the even quartic oscillator can be obtained from some periodic and doubly periodic surfaces by a sort of surgery.*

The correspondence between parameters A (asymptotic values, Nevanlinna parameters) and parameters a of the potential is very complicated, and the study of this correspondence is our main challenge. We have a map Ψ which assigns to a function $f \in G$ the parameter a of the differential equation (2). Telling something about critical values of this map is an important problem. Computer experiments show that the set of these critical values has a hexagonal lattice-like pattern. This pattern must be related to the doubly periodic structure that we see in the monodromy of the map Φ.

One can study the map Ψ asymptotically (see below), or one can hope to find points with special properties in the A-space such that the corresponding points in the a-space can be determined. There are obvious candidates of such special points in parameter space of cubic and even quartic potentials, and recent work of D. Masoero [56, 57] suggests that they are related to poles of special solutions of Painleve I and II equations which are called "integrales tritronquée". We are going to explore this connection.

Another long-standing problem is the study of "level crossing". We proved in [33] that all singularities of the multi-valued function $\lambda(a)$ for the even quartic family are algebraic ramification points, accumulating only to infinity. These ramification points are conjectured to be simple [11, 68]. We are going to investigate this. The critical values of the map Ψ above are contained in the set of values of parameter a where the level crossing occurs.

Especially interesting are spectral loci of certain real families of oscillators (2). A problem (2) is called PT-symmetric if the potential satisfies $P(-\overline{z}, a) = \overline{P(z, a)}, a \in \mathbf{R}$ and the rays L_1, L_2 are symmetric with respect to the imaginary axis. ("PT" stands for "parity and time" symmetry, but we don't discuss the physical interpretation here, referring to the papers of C. Bender [5, 6, 7]). The eigenvalues of a PT-symmetric problem are symmetric with respect to the real axis, and it is of great interest to physicists, when these eigenvalues are all real ("unbroken PT-symmetry") and how they escape from the real axis ("level crossing" or "breaking of PT-symmetry"). This motivates a study of spectral loci of real PT-symmetric families.

At this time, we have some preliminary results [35]. In this preprint, we consider the simplest PT-symmetric family of cubic oscillators

$$-y'' + (iz^3 + iaz)y = \lambda y, \quad y(\pm\infty) = 0. \tag{5}$$

It is known [66] that for $a \geq 0$, all eigenvalues are real. However, computer experiments show that for $a < 0$, pairs of adjacent eigenvalues collide and escape to the complex plane. The real spectral locus consists of a countable union of disjoint analytic curves Γ_n in the real (a, λ)-plane. For each $a \geq 0$ two eigenvalues lie on each curve. All these features observed previously in computer experiments are rigorously proved in [35]. The details of geometry of the spectral locus will be subject of further investigation. Our main finding for this family is the way of labeling the components Γ_n of the real spectral locus by the numbers of zeros of eigenfunctions that *do not* lie on the imaginary axis. This number is the "new quantum number" predicted by Trinh [70].

Next we propose to study in the similar way two 2-parametric PT-symmetric quartic families introduced in [8] and [9]. A preliminary study of the first family is in [34] but many features of the real spectral locus remain unproved. Especially interesting is the second family which (according to numerical evidence) displays all sorts of level crossing, with escape of eigenvalues to the complex plane and without. All these features are known only on the basis of computer experiments, and we wish to explain them and to prove them rigorously. This family has an additional interesting feature that it contains quasi-exactly-solvable (QES) subfamilies. This means that finitely many eigenfunctions are "elementary" in the sense that they have the form $p \exp q$ with polynomials p and q. The importance of QES families is due to the fact that their eigenvalues and eigenfunctions can be found exactly by solving algebraic equations. The QES part of the spectral locus makes a separate irreducible component of the whole complex spectral locus. In [34] we proved that this part is indeed irreducible, and we hope to be able to prove that the rest of the complex spectral locus is irreducible as well.

Our method of study of the real spectral loci consists of two parts. First part is the Nevanlinna parametrization described above. Second part is what we call "degeneration", that is finding the limit of the spectrum when one of the parameters tends to infinity. By a rescaling, this is equivalent to "singular perturbation". To be more precise, we consider the family of potentials

$$P_t(z, a, c) = tz^d + cz^m + p(z, a), \tag{6}$$

where $t \geq 0$, $c \in \mathbf{C}\backslash\{0\}$, and p is a polynomial of degree $m-1$. The normalization rays L_1 and L_2 being fixed, we study what happens to the spectrum as $t \to 0$. This is a general problem of singular perturbation which was extensively studied, see, for example, [11, 68, 17]. These authors use general theory of perturbation of linear operators [49]. Our method is different. We use the specifics of ordinary differential operators of second order with polynomial coefficients, and believe that analytic theory of such operators [67] will give more precise results for our problems. A preliminary study of this problem is contained in [35]. We proved that under certain conditions the limit of the discrete spectrum of (6) as $t \to 0$ is the discrete spectrum of P_0, or the eigenvalues of (6) escape to infinity, if the discrete spectrum of P_0 is empty. The assumptions which imply this are sufficient for our purposes in [35], but we know that they can be substantially relaxed. Finding the exact conditions when the limit of the spectrum spectrum of (6) coincides with the spectrum of the limit problem is subject of further investigation. The expected result is finding the limit of the discrete spectrum of (6) for almost all normalization rays L_1, L_2.

Our results on singular perturbation permit to prove rigorously that the spectrum and eigenfunctions of an even quartic can be approximated by those of the QES sextic, the fact conjectured by physicists long ago. Similarly, the spectrum and eigenfunctions of the PT-symmetric cubic can be approximated by those of the QES PT-symmetric quartic. This permits to reduce the study of the spectral loci of the even quartic and PT-symmetric cubic to purely algebraic problems. These algebraic problems have many interesting patterns (like the approximate hexagonal lattice symmetry mentioned above) which we will try to explain.

2. Some general problems of function theory related to the previous section

a) We plan to investigate the singularities of implicit functions $\lambda(a)$ defined by the general equations of the form (3) with entire F. It is known and easy to prove that these singularities can be either algebraic branch points or such that $\lambda(a) \to \infty$ as $a \to a_0$ along a curve. The set of singularities of the second type cannot be too large: *If $\lambda(a)$ is any germ at $a = a_0$ of the implicit function defined by (3), and $\gamma : [0, 1] \to \mathbf{C}$ any curve such that $\gamma(0) = a_0$, and $\epsilon > 0$ is given, then there exists a curve γ', $\gamma'(0) = a_0$ such that $|\gamma(t) - \gamma'(t)| \leq \epsilon$, and $\lambda(a)$ has an analytic continuation along γ'.* This result is due to Julia [48], and this is the only known general result on the subject. If F is of a special form $F(a, \lambda) = f(\lambda) - a$, then a more precise theorem of Gross holds: *Analytic continuation of the given germ can be performed along almost any ray from a_0.* It is known that the set of exceptional rays can have the power of continuum. We plan to study the following questions:

Question 1. *Does Gross's theorem hold for all implicit functions defined by entire relations (3)?*

Question 2. *Can one improve the exceptional set in Gross's theorem?*

Question 3. *For which classes of functions F one can assert that the implicit function defined by (3) has at most countable set of singularities?*

These questions are related to the study of analytic sets in $\mathbf{C}^n$ and their projections [29]. Question 3 might have a connection to the recent research in model theory [74, 76].

b) General properties of functions of class S. A meromorphic function f belongs to the class S if there is a finite set A such that (4) is a covering map. R. Nevanlinna and O. Teichmüller studied this class of functions in 1930-s from the point of view of value distribution theory [62, 75]. In [37] it was shown that S is a natural class for the study of transcendental holomorphic dynamics. As we have seen in section 1, these functions are also important in the study of spectral problems for polynomial potentials.

Two meromorphic functions f and g are called topologically equivalent if there are homeomorphisms ϕ and ψ such that $f \circ \phi = \psi \circ g$. It was shown in [37] that the set of meromorphic functions g topologically equivalent to a given $f \in S$ is a finite dimensional complex manifold M_f. The local parameters are the singular values A. These manifolds play an increasing role in holomorphic dynamics, and they parametrize the spectral loci (see section 1 of this proposal).

It is important to study compactification of these manifolds. This amounts to the following: function $g \in M_f$ varies holomorphically with the singular values (the points of the set A in (4)). What happens when two or more singular values collide? This problem arises both in holomorphic dynamics and in the "degeneration" procedure described in Section 1. One can show that the limiting function(s) always exist but they may be meromorphic only in a disc, rather then in the plane. Examples of this phenomenon will be described in the forthcoming paper of A. Eremenko and his former student S. Merenkov. The problem is to find good criteria which ensure that the limit functions are meromorphic in the plane. This will be a subject of our investigation.

Another question on class S is how restrictive is the condition $f \in S$ on a meromorphic function f? It is known that functions in S can have arbitrarily fast growth [58] but there is a universal estimate of the growth from below [30]. It is known that the shape of a tract of an entire function corresponding to the infinite asymptotic value can be almost arbitrary: for any unbounded simply connected domain D from which the point ∞ is accessible, one can find an non-constant entire function which is bounded outside D. Can this function f be chosen in the class S? This question arises in the recent research in holomorphic dynamics [64].

3. Asymptotic distribution of zeros of sequences of entire functions.

Let us denote by $N_\delta(f)$ the number of zeros of an entire function f in the region $\delta < |\arg z| < \pi - \delta$. In [12] we proved the following. Let f be a real entire function, with real zeros, but not in the Laguerre–Pólya class LP. Then there exist $\delta > 0$ and $\alpha > 0$ such that

$$\liminf_{n \to \infty} \frac{N_\delta(f^{(n)})}{n} \geq \alpha. \tag{7}$$

A theorem of Laguerre implies that $N_0(f^{(n)}) = O(n)$, so the rate of growth in our result is optimal. This was the final step in the proof of "Conjecture B" of

Pólya [63]. Complete proof of this conjecture is contained in the union of papers [53, 52, 12]. Our proof of (7) is non-constructive: it gives no specific values of α and λ.

We conjecture that α and λ in (7) can be explicitly estimated in terms of what we call the "Wiman genus" of f, which is defined as the smallest integer $p \geq 0$ in the representation $f = g(z)\exp(az^{2p+2})h_1(z)$ where g is a polynomial, $a \leq 0$ and h_1 is a canonical product of genus at most $2p+1$ with all zeros real. Every entire function of finite order with all but finitely many zeros real can be written in this form, so the Wiman genus is well defined. Functions of Wiman genus 0 with all zeros real form the class LP.

This conjecture seems very hard. It is related to the following general problem of potential theory which might be of independent interest. Suppose that three regions $\Omega_j,\ 1 \leq j \leq 3,\ \overline{\Omega_j} \subset \Omega_{j+1},\ 1 \leq j \leq 2$ are given, and let u be a harmonic function in Ω_2 which has no harmonic extension to Ω_3. Find a *subharmonic* function $\tilde{u}$ in Ω_3 such that $u(z) = \tilde{u}(z),\ z \in \Omega_1$, and such that

$$\int_{\Omega_3} \Delta \tilde{u}$$

is minimal, or estimate the last integral from below for all possible subharmonic extensions $\tilde{u}$.

For the beginning we plan to restrict the class of functions considered. One natural class consists of functions whose zeros have density on positive and negative rays. For these functions, we intend to compute not only the numbers α and δ in (7) but also the limit distribution of zeros of $f^{(n)}$ as $n \to \infty$.

The methods developed in [12] are flexible enough to handle other classes of functions and other linear operators besides differentiation. A general problem can be stated as follows: Let P_n be a sequence of linear operators, preserving a class of entire functions, and f a function in this class. Find the limit distribution of zeros of $P_n f$ as $n \to \infty$. This includes the following interesting cases that we plan to consider.

a) P_n is a sequence of operators which preserve polynomials with real zeros (see the survey [20] on such operators; they also preserve class LP), and f is an entire function with real zeros but not in the LP class. We expect to find out, for which sequences P_n the number of non-real zeros of $P_n f$ must tend to infinity.

For more restricted classes of functions f we hope to be able to find the asymptotic distribution of zeros of $P_n f$.

b) $P_n f$ is the n-th partial sum of the Taylor series of f, and f is admissible in the sense of Hayman [46], or belongs to some similar class. What can be said about the limit distribution of zeros of $P_n f$?

4. Polynomial approximation of sgn(x).

In [40], the following result is obtained. Let E_n be the error of the best uniform approximation to sgn(x) on the set $[-1,-a]\cup[a,1]$ by polynomials of degree n. Then

$$E_n \sim \frac{1-a}{\sqrt{\pi a n}}\left(\frac{1-a}{1+a}\right)^n. \tag{8}$$

The hardest part here is the precise constant $(1-a)/\sqrt{\pi a}$. Our proof was substantially simplified in [60].

Herbert Stahl suggested that we try to obtain a result of the same precision for two *asymmetric* intervals. It is convenient to normalize these intervals as $[-A,-1]\cup[1,B]$. This is a much harder problem. The similar problem for *rational* functions was completely solved by Zolotarev in 1877 and his solution has a wide range of applications. We expect that the solution of the polynomial problem will also be useful in applications, see, for example, [65, 25] for some applications.

A related problem of finding a polynomial of the least deviation from 0 on two intervals (=uniform approximation of z^n) was completely solved by N. Akhiezer [1]. His solution suggests that the asymptotic expression for E_n for asymmetric intervals must contain an oscillating factor. The result of Akhiezer was generalized by H. Widom [73] who replaced two intervals with an arbitrary finite system of smooth curves in the plane. In this generality, one cannot have an explicit result of the same precision as in [1]. For this reason we restrict ourselves to the simplest situation of two intervals, trying to obtain the most precise result.

The problem of approximating sgn(x) seems harder than approximating x^n, because sgn(x) is only piecewise analytic, rather than entire, and this creates substantial difficulties in applying the arguments of Akhiezer and Widom. W. Fuchs [43, 44] considered uniform approximation of piecewise analytic functions on several intervals of the real line, using the method of Widom, and obtained the limit $\lim_{n\to\infty}\log E_n/n$. We are aiming at a much more precise result, an asymptotic for E_n itself, rather than $\log E_n$.

Our method is based on a representation of the extremal polynomial in the form $\cos\phi(z)$ where ϕ is a conformal mapping of the upper half-plane or of a quadrant onto a so-called "comb domain". Functions of this form were intensively studied since the pioneering work of Marchenko and Ostrovskii [55]; their use in solution of extremal problems of approximation theory was proposed in [26].

The method consists of three steps: a) guessing the extremal polynomial in the form $\cos\phi(z)$, where ϕ is a conformal map onto an explicitly described domain, b) proving that this polynomial is extremal by using the general arguments of Chebyshev, and c) finding the asymptotic behavior of the conformal map ϕ as $n\to\infty$; this last step is usually technically hard.

The method also permits to study the best uniform approximation by entire functions of given exponential type σ, which is sometimes easier than approximation by polynomials.

For asymmetric intervals, we have already performed steps a) and b), and we hope to complete step c), which presents formidable technical difficulties, in the near future. A simple problem which we intend to solve first is the limiting case $B = +\infty$: one considers in this case the best uniform approximation of $\mathrm{sgn}(x)$ on $[-A, -1] \cup [1, +\infty)$ by entire functions of given exponential type σ.

To conclude this section, we mention that the initial goal of this project was to find some information on the mysterious "Bernstein constant". This is the constant μ in the asymptotics of the error $E_n \sim \mu/n$ of the uniform polynomial approximation of $|x|$ on $[-1, 1]$. This asymptotics was proved by Serge Bernstein in 1912, and he asked whether any kind of "explicit expression" for the constant μ exists. Bernstein conjectured that $\mu = (2\sqrt{\pi})^{-1}$ but this was refuted by Varga [71] who computed the first 50 digits of μ. For the modern account of the problem we refer to [54].

We derived (heuristically) a non-linear integral equation for the conformal map presumably related to the Bernstein problem. Bernstein's constant can be expressed in terms of the solution of this equation. To justify all this, one has to prove the existence and uniqueness of this solution. We hope to be able to do this.

In [41] we applied the same method of conformal mapping onto comb domains to solve an extremal problem about entire functions. *If f is an entire function of exponential type whose indicator diagram is $[-i\sigma, i\sigma]$, then the upper density of zeros of f is at most $c\sigma$, where $c = 1.508879...$ is the unique positive solution of the equation*

$$\log(\sqrt{c^2+1} + c) = \sqrt{1 + c^{-2}}.$$

This bound is exact.

Surprisingly, the same constant c appears in an old result of S. Bernstein on polynomial approximation of entire functions by polynomials, [15], [2, Anhang, III.83] which has no apparent relation to our result. We are going to investigate this intriguing coincidence.

5. Analytic invariant curves of rational functions and a functional equation.

The very last paragraph of the famous memoir of Fatou [42] that established the foundation of holomorphic dynamics begins with the words: *It remains to study analytic curves invariant under rational transformations, which are intimately related to the functions studied in this chapter. We will return to this subject soon.* (Our translation from the French). Fatou never returned to the subject, at least in his published work. The subject is challenging but almost nothing is known about it. We state the following

Conjecture 4. *Let γ be a simple closed analytic curve in the Riemann sphere, which is invariant under a rational function f. Suppose that γ is not contained in a Siegel disc or an Hermann ring of f, and that f is not a Lattés function. Then γ is a circle.*

In this statement we used the standard terminology of holomorphic dynamics, see, for example, [59].

The same conjecture under stronger assumption that γ is a "repeller" of f was proposed by F. Przytycki (private communication). We have no means to attack this conjecture in full generality. Let us consider the case when γ contains a repelling periodic point, and $f : \gamma \to \gamma$ is a covering of degree $m \geq 2$. Then we can prove the following. There exist rational functions g and h satisfying the functional equation

$$f \circ h = h \circ g \tag{9}$$

such that g has an invariant circle C and $h(C) = \gamma$. This functional equation (9) is of great independent interest [27, 45, 21]. Complete classification of its rational solutions is probably out of reach at this time, but there is a reasonable hope that one can prove, using this functional equation, that γ must be a circle. A counterexample would lead to a counterexample to Conjecture 4.

The result obtained in [13, 39] is related to Conjecture 4. *If the Julia set of a rational function lies on a smooth curve than it also lies on an (invariant) circle.* In connection with this result, it is interesting to classify all rational functions whose Julia set is contained in a circle. Surprisingly, this problem is hard (a "solution" given in [47] is incorrect as the examples in [39] show). Fatou [42] gave a complete classification of rational functions whose Julia set is a circle or an arc of a circle. Alternatively, the Julia set can be a Cantor subset of a circle, and there is no description of such functions. It is even not known whether there exists an algorithm which for a rational function with integer coefficients would determine whether the Julia set is contained in a circle. We plan to consider this problem, starting from functions of small degrees and trying to guess the pattern.

References

[1] N. Akhiezer, Über einige Funktionen, welche in zwei gegebenen intervallen am wenigsten von Null abweichen, I, II, III. Bull. Acad. Sci. URSS VII Ser., (1932) 9, 1163–1202; (1933) 3, 309–344; (1933) 4, 499–535.

[2] N. Achieser, Vorlesungen über Approximationstheorie, Akedemie-Verlag, Berlin, 1967.

[3] A. Avila and M. Lyubich, Regular or stochastic dynamics in real analytic families of unimodal maps. Invent. Math. 154 (2003), no. 3, 451–550.

[4] R. Bañuelos and T. Carroll, Sharp integrability for Brownian motion in parabola-shaped regions. J. Funct. Anal. 218 (2005), no. 1, 219–253.

[5] C. Bender, Making sense of non-Hermitian Hamiltonians, Rept. Progr. Phys., 70 (2007) 947.

[6] C. Bender, PT-symmetric versus Hermitian formulations of quantum mechanics, J. Phys. A, 39 (2006) 1657–1668.

[7] C. Bender, Introduction to PT-symmetric quantum mechanics, Contemp. Phys., 46 (2995) 277–292.

[8] C. Bender, M. Berry, P. Meisinger, Van M. Savage and M. Simsek, Complex WKB analysis of energy-level degeneracies of non-Hermitian Hamiltonians, J. Phys. A, 34 (2001) L31-L36.

[9] C. Bender and S. Böttcher, Quasi-exactly solvable quartic potential, J. Phys. A., 31 (1998) L273–L277.

[10] C. Bender and A. Turbiner, Analytic continuation of eigenvalue problems, Phys. Lett. A 173 (1993) 442–446.

[11] C. Bender and T. Wu, Anharmonic oscillator, Phys. Rev., 184 (1969) 1231–1260.

[12] W. Bergweiler and A. Eremenko, Proof of the conjecture of Pólya on the zeros of successive derivatives of real entire functions, Acta Math., 197 (2006) 125–146.

[13] W. Bergweiler and A. Eremenko, Meromorphic functions with linearly distributed values and Julia sets of rational functions, *Proc. AMS.* 137 (2009), 2329-2333.

[14] W. Bergweiler, A. Eremenko, J. Langley, Real entire functions with real zeros and a conjecture of Wiman, GAFA 13 (2003) 975–991.

[15] S. Bernstein, Limit laws of the theory of best approximations, Doklady Akad. Nauk, 58 (1947) 525-528.

[16] F. Berteloot et J. Duval, Sur l'hyperbolicité de certains complémentaires, Enseign. Math., 47 (2001) 253–267.

[17] E. Caliceti, S. Graffi and M. Maioli, Perturbation theory of odd anharmonic oscillators, Comm. Math. Phys., 75 (1980) 51-66.

[18] J. Clunie and W. Hayman, The spherical derivative of integral and meromorphic functions, Comm. Math. Helv., 40 (1966) 373–381.

[19] R. Conte and T. Ng, Meromorphic solutions of a third order nonlinear differential equation, arXiv:1002.1209.

[20] T. Craven and G. Csordas, Composition theorems, multiplier sequences and complex zero decreasing sequences, *Value distribution theory and related topics,* 131–166, Kluwer, Boston, MA, 2004.

[21] M. Dabija and M. Jonsson, Endomorphisms of the plane preserving a pencil of curves. Internat. J. Math. 19 (2008), no. 2, 217–221.

[22] E. Delabaere and F. Pham, Unfolding the quartic oscillator, Ann. Physics, 261 (1997) 180–218.

[23] E. Delabaere and F. Pham, Resurgent methods in semi-classical asymptotics, Ann. Inst. Poioncaré 71 (1999) 1–94.

[24] E. Delabaere and D. T. Trinh, Spectral analysis of the complex cubic oscillator, J. Phys. A, 33 (2000) 8771–8796.

[25] I. Diakonikolas, P. Gopalan, R. Jaiswal, R. Servedio, E. Viola Bounded Independence Fools Halfspaces, arXiv:0902.3757.

[26] A. Eremenko, Entire functions bounded on the real line, Soviet Math. Doklady, 37, 3 (1988) 693–695.

[27] A. Eremenko, Some functional equations related to the iteration of rational functions, Leningrad J. Math., 1, 4 (1990), 905-919.

[28] A. Eremenko, Meromorphic traveling wave solutions of the Kuramoto-Sivashinsky equation. Zh. Mat. Fiz. Anal. Geom. 2 (2006), no. 3, 278–286, 339.

[29] A. Eremenko, Exceptional values in holomorphic families of entire functions, Michigan J. Math., 54 (2006) 687–696.

[30] A. Eremenko, Transcendental meromorphic functions with three singular values, Illinois J. Math., 48 (2994) 701–709.

[31] A. Eremenko and A. Gabrielov, Rational functions with real critical points and the B. and M. Shapiro conjecture in real enumerative geometry, Ann of Math., 155 (2002), 105-129.

[32] A. Eremenko and A. Gabrielov, Degrees of real Wronski maps, Discrete and Computational Geometry, 28 (2002) 331–347.

[33] A. Eremenko and A. Gabrielov, Analytic continuation of eigenvalues of a quartic oscillator, Comm. Math. Phys., 287, (2009) 431–457.

[34] A. Eremenko and A. Gabrielov, Irreducivility of some spectral determinants, arXiv:0904.1714.

[35] A. Eremenko and A. Gabrielov, Singular perturbation of polynomial potentials in the complex domain with applications to PT-symmetric families, arXiv:1005.1696.

[36] A. Eremenko, A. Gabrielov, M. Shapiro and A. Vainshtein, Rational functions and real Schubert calculus, Proc. AMS, 134 (2006), no. 4, 949–957.

[37] A. Eremenko and M. Lyubich, Dynamical properties of some classes of entire functions, Ann. Inst. Fourier, 42 (1992) 1–32.

[38] A. Eremenko and D. Novikov, Oscillation of Fourier integrals with a spectral gap, J. Math. Pures Appl., 83 (2004) 313–365.

[39] A. Eremenko and S. van Strien, Rational functions with real multipliers, accepted in *Trans. AMS*.

[40] A. Eremenko and P. Yuditskii, Uniform approximation of sgn(x) by polynomials and entire functions, *J. d'Analyse Math.*, 101 (2007) 313-324.

[41] A. Eremenko and P. Yuditskii, Extremal problem for a class of entire functions, Comptes Rendus Acad, Sci., Parism 346 (2008) 825–828.

[42] P. Fatou, Sur les équations fonctionnelles (Troisième Mémoire), Bull. Soc. Math. France, 48 (1920) 208–314.

[43] W. Fuchs, On the degree of Chebyshev approximation on sets with several components, Izv. Akad. Nauk Armyan. SSR Ser. Mat. 13 (1978) 396-404, 541.

[44] W. Fuchs, On Chebyshev approximation on several disjoint intervals, in the book: Complex approximation (Proc. Conf. Quebec, 1978) 67-74. Birkhäuser, Boston, Mass., 1980.

[45] R. Grigorchuk and A. Zuk, The Ihara zeta function of infinite graphs, the KNS spectral measure and integrable maps. *Random walks and geometry,* 141–180, Walter de Gruyter, Berlin, 2004.

[46] W. Hayman, A generalisation of Stirling's formula. J. Reine Angew. Math. 196 (1956) 67–95.

[47] C. Inninger, Rational iteration. Characterizations of functions with empty Fatou set, Jordan arc Julia sets and with real Julia sets, Dissertation, Schriften der Johannes-Kepler- Universität Linz, Reihe C: Technik und Naturwissenschaften 35. Universitätsverlag Rudolf Trauner, Linz, 2001.

[48] G. Julia, Sur le domaine d'existence d'une fonction implicite définie par une relation entiére $G(x, y) = 0$, Bull. soc. math. Fr., 54 (1926) 26–37.

[49] T. Kato, Perturbation theory of linear operators, Springer, Berlin, 1966, 1976, 1980 etc.

[50] Critical points of real entire functions and a conjecture of Pólya, Proc. Amer. Math. Soc. 124 (1996), no. 3, 819–830.

[51] N. Kudryashov and M. Demina, Newton polygons for finding exact solutions, arXiv:nlin/0606018.

[52] J. Langley, Non-real zeros of higher order derivatives of real entire functions of infnite order. J. Anal. Math., 97 (2006), 357–396.

[53] B. Levin and I. Ostrovskii, The dependence of the growth of an entire function on the distribution of zeros of its derivatives, Sibirsk. Mat. Z. 1 (1960) 427–455.

[54] D. Lubinsky, On the Bernstein constants of polynomial approximation, Constr. Approx., 25 (2007) 303-366.

[55] V. Marcenko and I. Ostrovskii, A characterization of the spectrum of the Hill operator. Mat. Sb. 97(139) (1975), no. 4(8), 540–606, 633–634.

[56] D. Masoero, Poles of integrale tritronquee and anharmonic oscillators. A WKB approach, arXiv:0909.5537.

[57] D. Masoero, Y-system and deformed thermodynamical Bethe ansatz, arXiv:1005.1046.

[58] S. Merenkov, Rapidly growing entire functions with three singular values, Illinois J. Math. 52 (2008), no. 2, 473–491.

[59] J. Milnor, Dynamics in one complex variable, AMS, Providence, 2006.

[60] F. Nazarov, F. Peherstorfer, A. Volberg, P. Yuditskii, Asymptotics of the best polynomial approximation of $|x|^p$ and of the best Laurent polynomial approximation of $\mathrm{sgn}(x)$ on two symmetric intervals, Constr. Approx. 29 (2009), no. 1, 23–39.

[61] R. Nevanlinna, Über Riemannsche Flächen mit endlich vielen Winfungspunkten, Acta Math. 58 (1932) 295–373.

[62] R. Nevanlinna, Analytic functions, Springer, Berlin, 1970.

[63] G. Pólya, On the zeros of the derivatives of a function and its analytic character. Bull. Amer. Math. Soc., 49 (1943) 178-191.

[64] G. Rottenfuer, J. Rückert, L. Rempe and D. Schleicher, Dynamic rays of bounded-type entire functions, ro appear in Ann. Math., arXiv:0704.3213.

[65] A. Sherstov, Approximate inclusion-exclusion for arbitrary symmetric functions, Comput. Complexity 18 (2009), no. 2, 219–247.

[66] K. Shin, On the reality of eigenvalues for a class of PT-symmetric oscillators, Comm. Math. Phys., 229 (2002) 543–564.

[67] Y. Sibuya, Global theory of a second order linear ordinary differential equation with polynomial coefficient, North-Holland, Amsterdam, 1975.

[68] B. Simon, Coupling constant analyticity for the anharmonic oscillator, Ann. Physics, 58 (1970) 76–136.

[69] F. Sottile, Frontiers of reality in Schubert calculus. Bull. Amer. Math. Soc. 47 (2010), no. 1, 31–71.

[70] D.-T. Trinh, On the Sturm–Liouville problem for the complex cubic oscillator, Asympt. Anal., 40 (2004) 211-234.

[71] R. Varga, On the Bernstein conjecture in approximation theory. Constr. Approx. 1 (1985), no. 4, 333–348.

[72] S. Vernov, On elliptic solutions of the quintic complex one-dimensional Ginzburg-Landau equation, arXiv:nlin/0602060

[73] H. Widom, Extremal polynomials associated with a system of curves in the complex plane. Advances in Math. 3 (1969) 127–232.

[74] A. Wilkie, Some local definability theory for holomorphic functions. In: Model theory with applications to algebra and analysis, vol. 1 (Editors: Chatzidakis, Macpherson, Pillay and Wilkie, LMS Lect. Notes Sera 349, Cambridge UP, 2008.

[75] H. Wittich, Neuere Untersuchungen über eindeutige analytische Funktionen, Springer, Berlin, 1955, 1968.

[76] B. Zilber, Dimensions and homogeneity in mathematical structures. Connections between Model Theory and Algebraic and Analytic geometry. (Editor: A. Macintyre), Quaderni di metematica, vol. 6, Seconda Univ. di Napoli, 2000.