PROJECT SUMMARY

Alexandre Eremenko

Problems in geometric function theory

Submitting Organization: Purdue University

The proposer intends to continue his study of geometrical properties of meromorphic functions using a variety of analytic, geometric and topological methods, the approach which already brought significant results in his previous research. All problems in this proposal arise naturally from the results of the previous research of the proposer funded by NSF.

Proposed research will expand our understanding of solutions of transcendental equations that occur in analysis, mathematical physics and control theory.

The broader impact of this activity is two-fold:

1. The results of geometric function theory are used in many areas of applied mathematics from mathematical physics to computer science. The choice of the problems in this proposal is motivated by their intrinsic, pure mathematical interest. However the previous work of the proposer on this subject already found applications, sometimes unexpected, in control theory, material science, computer science, signal processing, mathematical physics and astrophysics, and it is expected that the proposed research will have similar applications.

2. The proposed activities will have an impact on education and educator development: the results of this research will be used in graduate courses; graduate students and post-doctoral researchers will be involved in the research. The proposer will also continue his efforts in creating public resources which are intended to increase public scientific literacy and public engagement with science and technology. This includes his web pages and active participation in the projects like Math Overflow.

The proposer plans to concentrate on the following concrete problems of function theory:

The study and classification of metrics of constant positive curvature with conic singularities on the sphere. This geometric problem is related to the study of Heun's equation. Solutions of Heun's equation lie on the boundary of the set which is called "Special functions of mathematical physics": they frequently occur in the problems of mathematical physics, but much less is presently known about them then about most other special functions.

The study of a new kind of extremal problems for analytic functions in the unit disc which are relevant for control theory.

The study of analytic invariant curves for rational functions. This is related to holomorphic dynamics and classical functional equations.

Continuation of the study of the so-called Brody curves, which are holomorphic curves in the projective space with a uniform bound on their length distortion.

RESULTS FROM PRIOR NSF SUPPORT

A. Eremenko was supported (jointly with co-PI, A. Gabrielov) by the NSF grant DMS–1067886 (funding period: June 1, 2011 – May 31, 2014). The research was titled "Meromorphic functions and their applications"; the amount of support was $310,000. The results of this research are contained in the 15 papers listed below (12 published and 3 accepted for publication). In addition, one preprint is posted on the arXiv but not submitted to a journal.

1. (with A. Gabrielov) *Singular perturbation of polynomial potentials in the complex domain with applications to spectral loci of PT-symmetric families*, Moscow Math. J., 11, 3, (2011) 473-503.
2. (with A. Gabrielov) *Quasi-exactly solvable quartic: elementary integrals and asymptotics*, J. Phys. A: Math. Theor. 44 (2011) 312001
3. (with A. Gabrielov) *Quasi-exactly solvable quartic: real algebraic spectral locus*, J. Phys. A: Math. Theor. 45 (2012) 175205.
4. (with A. Gabrielov) *Two-parametric family of PT-symmetric quartics*, J. Phys. A: Math. Theor. 45 (2012) 175206.

This is a set of closely related papers dedicated to the study of spectral loci of anharmonic oscillators. We consider boundary value problems of the form

$$-y'' + P(z, a)y = \lambda y, \quad y(z) \to 0, \quad z \in L_1 \cup L_2,$$

where P is a polynomial in z depending holomorphically on a parameter a, and L_i are two rays in the complex plane. With an appropriate choice of L_i, the problem has discrete sequence of eigenvalues λ which satisfy an equation $F(\lambda, a) = 0$ with an entire function F. The set of zeros of F is called the spectral locus. We study topology and geometry of the spectral locus, and especially its real part. In the first paper we develop a singular perturbation theory for such problems, which is then applied to concrete cases of interest to physicists when P is a polynomial of degree 3 or 4 in z (anharmonic oscillators). In the other three papers we study the most interesting and difficult case of the quartic PT-symmetric oscillator

$$-y'' + (z^4 - 2az^2 + 2Jz)y = \lambda y, \quad y(te^{\pm\pi i/3}) \to 0, \quad t \to +\infty.$$

When J is a positive integer, this problem is quasi-exactly solvable (QES) which means that the spectral locus is reducible and has an algebraic component. This algebraic component is studied in detail in papers 2, 3. In the last paper 4, we give a parametrization of the whole spectral locus, considering both a and J as continuous parameters. These papers prove many essential properties of the spectral locus, some of them discovered earlier by physicists by numerical computation. For example, we rigorously proved the existence of "crossing points" on the real part of the spectral locus. In paper 2, a new interesting identity related to integration in closed form is discovered. This identity was later proved in [49]. Paper 4 was selected by J. Phys. A for the list of "Highlights of 2012".

5. (with P. Yuditskii) *Polynomials of the best uniform approximation to sgn(x) on two intervals*, J. d'Analyse math., 114 (2011) 285-315.

This is a continuation of our previous work [27] in which we found the best polynomial approximation of $\mathrm{sgn}(x)$ on two symmetric intervals. W. Hayman and H. Stahl asked whether this result can be extended to arbitrary two intervals whose union does not contain 0. This problem turned out to be much harder, and the answer is more complicated, but we did overcome all difficulties, and obtained an exact asymptotic formula for the error term, together with a qualitative description of the extremal polynomials. The new feature is that the error term oscillates as degree varies. The error term is expressed in theta-functions. The result in [27] found several unexpected applications in computer science [13, 64, 65]. We expect that the new, more general result will find similar applications.

6. (with S. Ivanov) *Spectra of the Gurtin-Pipkin type equations*, SIAM J. on Math. Anal., 43, N5 (2011) 2296–2306.

In this paper, we apply geometric function theory to a problem of applied mathematics, description of the spectra of certain integro-differential equations arising in the study of heat transfer in media with finite propagation speed, systems with thermal memory, viscoelasticity problems and acoustic waves in composite media. We obtain exact estimates for the non-real part of the spectrum.

7. (with M. Barrett) *A generalization of a theorem of Clunie and Hayman*, Proc. Amer. Math. Soc., 140, (2012) 1397-1402.

Theorem of Clunie and Hayman gives an estimate of the growth of an entire function in terms of the growth of its spherical derivative. In this paper of PI and his graduate student, this theorem is generalized to holomorphic curves in complex projective space. Earlier, the PI obtained such a generalization, but the proof was non-constructive, and did not give explicit estimates. Paper 7 makes all estimates explicit. The role of spherical derivative is played by what we call the Fubini–Study derivative. It measures the length distortion by the curve from the Euclidean metric in the plane to the Fubini–Study metric in the projective space. This research was mainly motivated by the so-called Brody curves, whose Fubini–Study derivative is bounded. Brody curves play a central role in the theory of mean dimension of Gromov and Lindenstrauss [14]. Recently, the methods of this paper had an unexpected development in the work of B. da Costa and J. Duval [12] who made a remarkable conjecture about a version of the Second Main Theorem of Cartan for Brody curves. They proved several special cases, and a weak form of this conjecture. During da Costa's stay at Purdue as a post-doctoral scholar, we tried to prove the full conjecture, without much success so far. We plan to continue this investigation. See the project description for more detail.

8. (with W. Bergweiler) *A property of the derivative of an entire function*, Ann. Acad. Sci. Fenn., 37, (2012) 301-307.

This is a new general result on arbitrary entire functions, conjectured by A. Weitsman who was motivated by the theory of minimal surfaces. The theorem says that for every non-constant entire function f, the derivative f' is unbounded on the full f-preimage of any unbounded set.

9. *Invariant curves and semiconjugacies of rational functions*, Fundamenta Math., 219, 3 (2012) 263-270.

This paper addresses an old problem stated by P. Fatou in the end of his famous memoir on functional equations [28] which is considered a starting point of modern holomorphic dynamics. The question is: which Jordan analytic curves, besides circles, can be invariant under a rational function. The paper contains several results in this direction. One of the main results can be stated as follows: If the restriction of the function on the curve is not injective, then the curve must be algebraic, unless the function belongs to a narrow explicitly described exceptional class. Soon after the paper was published, Peter Müller [52] constructed first non-trivial examples of such invariant curves (other than circles). More detail on this is given in the project description.

10. (with P. Yuditskii) *Comb functions*, Contemp. Math., 578 (2012) 99-118.

This is a survey paper on some classes of entire functions and conformal maps which frequently arise in the research on approximation theory and spectral theory. The conformal maps studied in this paper map the upper half-plane onto regions D with the property that for every point $a \in D$, the vertical ray $a+it,\ t \geq 0$ is contained in D. The simplest regions of this sort are obtained from the plane or from the upper half-plane by making some vertical cuts. This explains the names “comb regions” and “comb functions”. Several important classes of entire functions, like the Laguerre–Pólya class have a geometric description in terms of comb regions. This geometric description helps in solving extremal problems of approximation theory, and it was frequently used in our previous work with P. Yuditskii.

11. (with E. Lundberg) *Non-algebraic quadrature domains*, Potential Analysis, 38, 3 (2013), 787-804.

A quadrature domain D is characterized by the existence of a quadrature formula

$$\int_D u(x)dx = \sum_{k=1}^{n} a_k u(x_k),$$

which holds for every harmonic function integrable in D, with points x_k and constants a_k depending only on D. Quadrature domains are intensively studied nowadays in both pure and applied mathematics [73]. In dimension 2, the boundary of a quadrature domain is always an algebraic curve. In this paper, we construct the first examples of quadrature domains in $\mathbf{R}^4$ whose boundaries are not algebraic surfaces. This answers a question of H. S. Shapiro and B.

Gustafsson. It is not known whether such non-algebraic quadrature domains exist in $\mathbf{R}^3$ or in any $\mathbf{R}^n$ with odd n. Quadrature domains are closely connected with models of Laplacian growth, and our paper gives new integrable examples of Laplacian growth in $\mathbf{R}^4$.

12. (with W. Bergweiler) *Goldberg's constants*, J. d'Analyse Math., 119, 1, (2013) 365-402.
13. *Simultaneous stabilization, avoidance and Goldberg's constants*, arXiv:1208.0778.

In these papers the following extremal problem is considered. Let F be the class of an analytic function in the unit disc, for which the equations $f(z) = 0$ and $f(z) = 1$ have finite, non-zero *distinct* numbers of solutions (counting multiplicity). How small can be the radius of a hyperbolic disc containing all these solutions? This problem was considered by Goldberg [31], who proved that there is a positive universal constant which bounds this radius from below. Much later it was found that this result is relevant for control theory [5]. It turns out that the problem is also related to the "avoidance problem" studied in [44]. In paper 12, we completely solved the analogous problem for meromorphic functions, which is also relevant for control theory, and gave upper and lower estimates for what we call Goldberg's constant. We also found a function which is conjectured to be extremal, and proved its extremality in a similar, more restricted problem. Paper 13 is a survey of the known results on this problem. More detail is given in the project description.

14. *Normal holomorphic maps from a cylinder to a projective space*, Accepted in Houston J. Math., arXiv:1208.0779.

This paper contains an answer on a question of M. Tsukamoto on generalization of the classical result of A. Ostrowski [54] on the parametric description of normal meromorphic functions in $\mathbf{C}^*$. "Normal" means that the family $\{f(\lambda z) : \lambda \in \mathbf{C}^*\}$ is a normal family in the sense of Montel. In paper 14, Ostrowski's result is generalized to holomorphic curves from $\mathbf{C}^*$ to a complex projective space. A parametrization of normal curves $\mathbf{C}^* \to \mathbf{P}^n$ in terms of zeros of their coordinates is obtained. This result can be considered a preliminary step to parametrization of Brody curves.

15. K. Ramachandran, *Asymptotic behavior of positive harmonic functions in certain unbounded domains*, Accepted in Potential Analysis, arXiv:1211.2214.

This paper is based on the thesis of K. Ramachandran who is a PhD student of PI. This is an attempt to prove a multi-dimensional analog of Warshawski distortion theorem. Let D be an unbounded region in $\mathbf{R}^n$, and u a positive harmonic function in D, zero on the boundary. The problem is to relate the growth of u at infinity to the geometric properties of D. A precise lower estimate of the growth is given by Carleman's inequality for arbitrary domains. The question is how to obtain an upper estimate, even for a very restricted class of domains, say paraboloids of revolution. In dimension 2, this was done by Warshawski using conformal mapping. Paper 15 contains some extirpates of this sort in arbitrary dimension, generalizing the previous results of Hayman and Carroll for paraboloids. The result has applications in probability theory [2].

16. (with M. Barrett) On the spherical derivative of a rational function, Submitted for publication, arXiv:1207.5214.

This paper addresses the problem inspired by the work of Gromov and d'Ambra [1] and its development in [6, 7, 58]. Let f be a rational function, and $K(f) = \max_z \|f'\|(z)$, where $\|f'\|$ is the norm of the derivative with respect to the spherical metrics in the domain and in the target. What is the minimum of $K(f)$ over all rational functions of given degree d? A simple argument gives that $K(f) \geq \sqrt{d}$.

Our first result shows that there exist absolute constants $C > c > 1$, such that this minimum is between $c\sqrt{d}$ and $C\sqrt{d}$ for $d > 2$. Then we study the asymptotic behavior of $K(f^{*n})$ as $n \to \infty$ for the iterates f^{*n}. An unexpected conclusion is that Lattè's functions are not extremal for this last problem. For the definition of Lattès functions, see project description.

Broader impact of the PI's activities

Human resources development. In the period 2010-2014 A. Eremenko had two PhD students, both started in 2009.

Matthew Barrett worked on holomorphic curves in projective space. He defended his PhD in 2012. The main results of his thesis are published in the papers 7, 16 described above, both joint with the PI.

Koushik Ramachandran is scheduled to defend his thesis in spring 2014. The main result of his thesis is in the paper 15, which has been accepted for publication in the journal Potential Analysis.

In the same period, A. Eremenko advised two post-doctoral fellows, Eric Lundberg and Bernardo da Costa. There is one joint publication, 11 of the PI and E. Lundberg. Lundberg will stay at Purdue for another year, and the PI is planning to continue this collaboration.

In summer 2013, the PI was a member of the examination committee of Per Alexandersson, a PhD student of the co-PI, A. Gabrielov and B. Shapiro. Alexandersson defended his PhD thesis in the University of Stockholm. Alexandersson's thesis was a part of the previous project of the PI funded by NSF.

Enhancing infrastructure for research and education and increasing public scientific literacy and public engagement with science and technology

The PI maintains a web page, www.math.purdue.edu/˜eremenko which contains various resources like lists of unsolved problems for beginning researchers, some popular and advanced lectures and essays on mathematical, scientific and engineering subjects by the PI, list of free recourse of mathematical literature, and so on. Some materials from this web page were used in the scientific programs of the mass media, like the radio broadcast in the series "Engines of our Ingenuity", http://www.uh.edu/engines/epi2703.htm.

The PI is active in the mathematical web site MathOverflow.

PROJECT DESCRIPTION
Problems in geometric function theory

1. Metrics of constant positive curvature with conic singularities on the sphere

The PI plans to work on this part of the proposal jointly with A. Gabrielov (Purdue U.) and V. Tarasov (IUPUI).

Conformal metrics of constant curvature K with conic singularities on a Riemann surface S are described by densities ρ which satisfy the equation

$$\Delta \log \rho + K\rho^2 = 0, \quad \text{on} \quad S\backslash\{a_1, \ldots, a_n\}, \tag{1}$$

and $\rho(z) \sim |z|^{\alpha_j - 1}$ for the local coordinate z which is 0 at a_j. Geometrically this means that the point a_j is a conical point with the total angle $2\pi\alpha_j > 0$. The question is whether such a metric with prescribed a_j and α_j exists, and if so, whether it is unique.

When $K \leq 0$, a complete answer to this question is known since the times of E. Picard [56, 57]; for a modern proof we refer to [69]: *For $K \leq 0$, such a metric exists if and only if the restriction that follows from the Gauss–Bonnet theorem is satisfied. This metric is unique for $K < 0$ and unique up to a constant multiple for $K = 0$.*

From now on we restrict ourselves to the simplest unsolved case when S is the sphere and $K = 1$. Gauss–Bonnet theorem gives for this case the necessary condition of existence

$$2 + \sum_{j=1}^{n} (\alpha_j - 1) > 0. \tag{2}$$

The strongest known general result on the question can be found in [69] and [45]: *if in addition to (2) the inequality*

$$2 + \sum_{j=1}^{n} (\alpha_j - 1) < 2\min\{1, \min_{1\leq j\leq n} \alpha_j\}, \tag{3}$$

is satisfied, then the metric in question exists and is unique.

Condition (3) is certainly not necessary, as the following two other known cases show:

1. The case of at most 3 singularities [70, 72, 18, 29].
2. The case when all α_j are integers [32, 24, 25, 61, 26].

These known results also show that in general, the Gauss–Bonnet condition is not sufficient for existence, and the metric in question is not necessarily unique.

Besides the great intrinsic interest, the question has many connections with other areas of mathematics and physics [15, 16, 36, 29, 68]

An interesting and important case is the symmetric version of the problem: *assuming that the points $a_1, \ldots, a_n$ lie on the real line, to determine existence and uniqueness of the* *metrics symmetric with respect to the real line.* Assuming that ρ is such a symmetric metric we can cut S along the real line, and each of the pieces will become a *spherical polygon.* So the symmetric case of the question is equivalent to the question of existence and uniqueness of a spherical polygon (a surface, not a curve!) with prescribed angles and prescribed images of vertices under a conformal map onto the upper half-plane.

Notice that for $K = 0$ the answer is given by the Christoffel–Schwarz formula, thus the symmetric version of our problem can be considered as a generalization of this formula to spherical polygons.

The main result of [24, 25] can be restated in the following way: *If all α_j are integers, and a_j lie on the real line, then all resulting metrics are symmetric, up to a simple transformation.* This is very non-trivial because the metric is not unique.

We are planning to investigate whether this result extends to general, non-integer α_j.

In view of the known difficulty of the problem, we plan to concentrate our study of the general and symmetric problems to the simplest unsolved case $n = 4$. The symmetric problem with $n = 4$ is essentially equivalent to the classification of spherical geodesic quadrilaterals up to an isometry.

By a *spherical quadrilateral* we mean a simply connected surface homeomorphic to the closed disc with 4 marked boundary points, equipped with the metric of constant curvature 1, and such that the 4 sides are geodesic with respect to this metric. We wish to classify such objects up to an isometry.

Similar problems were studied a lot in the end of 19-th and beginning of 20-th century [35, 63, 37, 38] but a complete classification was not obtained.

It seems that the methods based on PDE and Calculus of Variations [56, 57, 69] do not give the result when inequality (3) is violated. We intend to study the problem using the two other available methods, both going back to F. Klein [41, 42]. The so-called *developing map* f associated with (1) is a multi-valued analytic function which satisfies a Schwarz differential equation, so $f = y_1/y_0$, where y_0 and y_1 are two linearly independent solutions of a linear differential equation. In the case of 4 singularities, this linear differential equation is equivalent to the Heun equation

$$y'' + \left(\sum_{j=1}^{3} \frac{1-\alpha_j}{z-a_j}\right) y' + \frac{Az-\lambda}{(z-a_1)(z-a_2)(z-a_3)} y = 0, \tag{4}$$

where the singular points are a_1, a_2, a_3 and ∞, and the parameter A is related to the data $\alpha_j,\ 1 \leq j \leq 4$ as follows:

$$A = \alpha'\alpha'', \quad \sum_{j=1}^{3} \alpha_j + \alpha' + \alpha'' = 2, \quad \alpha_4 = \alpha' - \alpha''.$$

The number λ is called the *accessory parameter*, and our problem is equivalent to the following: *for given a_j, α_j, to determine λ in such a way that the monodromy group of the equation*

(4) is a subgroup of the unitary group $SU(2)$. Notice that the classical case $K < 0$ can be reduced to a similar problem about Heun's equation with $SL(2, \mathbf{R})$ instead of $SU(2)$.

In this form, the symmetric problem (when a_j are real) can be reduced to several eigenvalue problems, parameter λ playing the role of eigenvalue. However these eigenvalue problems are not self-adjoint and very singular, and the usual Sturm–Liouville theory does not apply as it was applied in the case $K < 0$ in the classical papers [42, 35, 66, 67]. We hope to be able to use the modern development in [50, 51, 30, 43] to treat these eigenvalue problems.

Eigenvalue problems of the considered type usually have finitely many non-real eigenvalues which suggests that Hermitian operators in a space with indefinite metrics are involved [30, 43].

In some cases, the equation for the eigenvalue λ happens to be algebraic, and we hope to use the powerful new methods developed in [50, 51] in these cases.

We plan to apply these methods to both the general and the symmetric problem.

Another approach to the symmetric problem, also going back to Klein, is purely geometric. The developing map f restricted to the upper half-plane H is a local homeomorphism from H to the sphere, and the image of the real line is contained in 4 great circles. These 4 circles make a relatively simple cell decomposition of the sphere, and we pull-back this cell decomposition to H. The problem of classification of spherical quadrilaterals can be reduced in this way to a purely combinatorial problem of classification of certain cell decomposition of the topological disc H. Klein succeeded in classification of spherical triangles in this way [41]. Some other papers where this approach was used are [37, 72, 76], but quadrilaterals are treated only in [37], and the results are incomplete.

It is this method that permitted us to prove the 1-dimensional case of the B. and M. Shapiro conjecture, [24, 25] which is a special case of the problem considered here, when the α_j are integers. Later, the B. and M. Shapiro conjecture was proved in the full generality in [50, 51] by reducing it to an eigenvalue problem for a symmetric matrix.

In the proposed research we are planning to combine the analytic and the geometric methods described above.

2. Goldberg's constants and similar extremal problems

Let F_0 be the class of all holomorphic functions f defined in the rings $\rho(f) < |z| < 1$, omitting 0 and 1, and such that the curve

$$\gamma(f) = f\left(\frac{1+\rho(f)}{2}\exp(it)\right), \quad 0 \leq t \leq 2\pi$$

has non-zero, distinct indices with respect to 0 and 1.

Let F_1, F_2, F_3, F_4 be subclasses of F_0 consisting of , meromorphic, rational, holomorphic and polynomial functions in the unit disc $\mathbf{U}$, respectively. So all zeros, poles and 1-points of these functions lie in the disc $|z| < \rho(f)$.

We define the constants

$$A_j = \inf\{\rho(f) : f \in F_j\}, \quad 0 \le j \le 4.$$

A. A. Goldberg [31] proved that

$$0 < A_0 = A_1 = A_3 < A_2 = A_4 < 0.0319.$$

So the problem is to find or estimate A_0 and A_2. We call these two constants Goldberg's constants.

Much later, it was found that Goldberg's problem is closely related to the following two problems.

Strong interpolation question [5] *Given three non-negative divisors of finite degree in the unit disc, when is it possible to find a meromorphic function f in $\mathbf{U}$ whose divisors of zeros, poles and 1-points coincide with these three divisors?*

We notice that a similar problem was proposed by R. Nevanlinna for meromorphic functions in the plane and divisors of infinite degree [53]. It is easy to give many necessary conditions, but research on the Nevanlinna problem shows that probably no reasonable verifiable sufficient conditions can be given.

Unlike for Nevanlinna problem, the expected answer to the Strong interpolation question with finite degree divisors in the unit disc is expected to have the answer in the form of inequalities. This is confirmed by a partial result in [4].

Avoidance problem [44, 5] *Given three rational functions ϕ_j, $1 \le j \le 3$, when is it possible to find a meromorphic function g in $\mathbf{U}$, such that the graph of g in $\mathbf{U} \times \overline{\mathbf{C}}$ is disjoint from the graphs of ϕ_j.*

It turns out that the Strong interpolation problem is (almost) equivalent to the Avoidance problem, and that Goldberg's theorem gives the only known general restrictions on solvability of both problems [5, 20].

Moreover, Avoidance problem is equivalent to an important problem in Control theory, namely the problem of simultaneous stabilization of three plants by one controller [5].

All this justifies our interest in the study of Goldberg's constants A_0 and A_2. Constant A_0 was recently found in [4]:

$$A_0 = \exp\left(-\frac{\pi^2}{\log(2+2\sqrt{2})}\right) \approx 0.0037.$$

Concerning A_2, we made a conjecture in [4] about the extremal function,and obtained some partial results, including the estimates

$$0.00587 < A_2 \le 0.0252896,$$

where the right hand side is the conjectured extremal. The conjectured extremal constant and extremal function cannot be simply expressed in terms of known constants or elementary functions. The conjectured extremal function is actually a composition of elliptic modular functions and a solution of a special Heun equation, with special choice of parameters.

One promising approach to proving our conjecture about A_2 is based on variation of critical values of a meromorphic function in the unit disc. This kind of variation was considered by Goodman [33] but the method needs further development to give the required result.

Several similar problems are mentioned in [20, 4]. One of them is a simplified version of the famous "Belgian chocolate problem" by V. Blondel [5]: *Let f be a real holomorphic function in the unit disc, which has exactly one simple zero at* 0*, and the equation $f(z) = 1$ has exactly* 2 *solutions z_1, z_2, counting multiplicity. What is the minimal possible value of* $\max\{|z_1|, |z_2|\}$*?*

This seems to be the simplest problem of the type that we consider here, and we hope to be able to solve it with a variational method.

Even approximate, numerical solution of these problems are relevant for control theory, but they are notoriously difficult to obtain [77].

In control theory, one is mostly interested in *rational* solutions of the two problems mentioned above. Goldberg's theorem gives a universal restriction on rational solutions which holds independently of the degree. Our conjectures about the extremal function lead to an interesting conjecture about extremal functions in the class of rational functions of fixed degree:

Conjecture. *In the class of of rational functions of fixed degree, the extremal function for A_2 is a Belyi function, which means that it has only* 3 *critical values,* 0*,* 1 *and* ∞*.*

The name Belyi functions comes from the famous theorem of Belyi [62] about functions on compact Riemann surfaces which have only three critical values. Belyi functions are intensively studied from various points of view, mainly because of their connection to number theory. Belyi rational functions are determined up to an affine change of the independent variable by a purely combinatorial object, a triangulation of the sphere. If our conjecture is true, this will show that Belyi functions are relevant for control theory, and the existing algorithms for their computation can be used to solve problems of control theory. The upper estimate of A_2 obtained by A. A. Goldberg is based on computation of a Belyi function.

3. Analytic invariant curves of rational functions and a functional equation.

The very last paragraph of the famous memoir of Fatou [28] that established the foundation of holomorphic dynamics begins with the words: *It remains to study analytic curves invariant under rational transformations, which are intimately related to the functions studied in this chapter. We will return to this subject soon.* (PI's translation from the French). Fatou never returned to the subject, at least in his published work. The subject is challenging but almost nothing was known about it until the publication of [17].

To explain the meaning of this problem, we recall one of the main results of Fatou, in the special case that the Julia set of a rational function f is a Jordan curve which is the common boundary of two immediate basins of attracting fixed points. For example, this is the case when $f(z) = z^2 + c$ with $|c|$ sufficiently small. This curve is invariant.

Fatou proved that this curve either has no tangent at any point (and not rectifiable), or it is a circle.

This was the beginning of the enormous area of research on "fractals" in holomorphic dynamics.

Fatou's proof consists of two parts. First part proves that if the invariant curve has a tangent at some point, or is rectifiable, then it must be analytic. The second part is the proof that an analytic invariant curve must be a circle.

The first part uses only local considerations (in a neighborhood of a curve), and it has been substantially generalized in many papers of the modern period. One of the most powerful generalizations considers holomorphic maps defined only in a neighborhood of the invariant curve, and prove under certain conditions, that the invariant curve is either analytic or has Hausdorff dimension strictly greater than 1 [59, 60].

The second part of Fatou's argument is of global nature; it uses substantially the fact that f is a rational function, and is based on the study of rational solutions of certain functional equations.

It is this second part which Fatou proposed to generalize to Jordan analytic invariant curves, without the assumption that these curves are the boundaries of the invariant domains.

A similar question was asked by Przytycki: *Are there Jordan analytic repellers for rational functions which are not circles?*

A partial answer is given by the following

Theorem. [17] *Let f be a rational function, and γ a Jordan analytic f-invariant curve, such that the restriction of f on γ is not a homeomorphism, and γ contains a repelling fixed point of f.*

Assume in addition one of the following:

(i) γ belongs to the Julia set of f, or

(ii) there are no critical points and no neutral rational fixed points of f on γ.

Then γ is an algebraic curve, unless f is a Lattès function. Moreover, if f is not a Lattès function, then there exists a real rational function g and a rational function h, such that γ contains $h(\mathbf{R})$, and the functional equation

$$f \circ h = h \circ g \tag{5}$$

holds.

We recall that Lattès functions are those rational functions f, which arise in multiplication theorems for elliptic functions F

$$F(kz) = f \circ F(z), \quad |k| > 1.$$

Latteès functions are exceptional in many problems of holomorphic dynamics [48]. They can have both algebraic and transcendental invariant curves of the type described in Theorem 1.

Equation (5) is called a semiconjugacy equation; it was studied at least since the times of Julia [40], and it arises in several areas of mathematics [34, 11, 55]. It is interesting that all solutions with transcendental h have been completely described [40, 19], but the rational case seems much harder [55].

Until recently, there were no non-trivial examples of curves (other than circles) satisfying the conditions of Theorem 1. Such examples were recently constructed by Peter Müller, and his construction shows that they are abundant.

The questions which arise in connection with Theorem 1 are the following.

1. Can conditions (i) and (ii) be removed? They seem to be technical assumptions needed in the proof, and the PI believes that one can get rid of them. If not, some new type of invariant analytic curves will be discovered.

2. Can one generalize Theorem 1 to curves which are not necessarily Jordan?

3. Can one give some classification of rational solutions of equation (5) relevant to Theorem 1? That is to classify all triples of rational functions satisfying (5) such that g is real but $h(\mathbf{R})$ is not contained in a circle on the Riemann sphere.

The most challenging question is 3. Pakovich [55] recently obtained certain classification of all possible solutions of (5) but the PI was unable so far to obtain from this classification any information on question 3.

The ultimate goal is to give some classification of all possible analytic curves invariant under rational functions.

Theorem 1 and the proposed research deals with those with those invariant curves γ for which the restriction of f on γ is not a homeomorphism.

There are well-known examples of Jordan analytic invariant curves which are mapped by f homeomorphically. Such curves occur in Siegel discs and Hermann rings. However, the following question seems very difficult:

Can a rational function have a Jordan analytic invariant curve which is mapped onto itself homeomorphically, and which belongs to the Julia set? How many such curves for a given rational function may exist?

4. Second Fundamental Theorem for Brody curves

A holomorphic curve is a holomorphic map f from the complex line $\mathbf{C}$ to the complex projective space $\mathbf{P}^n$. One can write such a map in homogeneous coordinates $f = (f_0 : \ldots : f_n)$, where f_j are entire functions with no zeros common to all of them. These f_j are defined up to a common factor which can be any zero-free entire function. The Nevanlinna–Cartan characteristic $T(r, f)$ can be defined by

$$T(r, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|f\|(re^{it})dt - \log \|f\|(0).$$

Here $\|f\| = \sqrt{|f_0|^2 + \ldots + |f_n|^2}$, and one can easily check that the definition of $T(r, f)$ and other definitions in this section are independent of the choice of a homogeneous representation. If L is a hyperplane with a homogeneous equation $a_0 w_0 + \ldots + a_n w_n = 0$, we consider the counting function $n(r, L, f)$ which is the number of zeros of the entire function $g_L = a_0 f_0 + \ldots + a_n f_n$ in the disc $|z| \leq r$, counting multiplicity, and define the Nevanlinna counting function by the averaging:

$$N(r, L, f) = \int_0^r (n(t, L, f) - n(0, L, f)) \frac{dt}{t} + n(0, L, f) \log r.$$

With this notation, the Second Fundamental Theorem of Cartan implies that for every curve f whose image is not contained in any hyperplane, and for any hyperplanes $L_1, \ldots, L_q$ in general position, we have

$$(q - n - 1)T(r, f) \leq \sum_{j=1}^{q} N(r, L_j, f) + O(\log rT(r, f)), \quad r \to \infty, \quad r \notin E,$$

where $E \subset (0, \infty)$ is a set of finite length.

A holomorphic curve is called a *Brody curve* if the Fubini–Study derivative,

$$\|f'\| = \|f\|^{-2} \left(\sum_{i<j} |f_i' f_j - f_i f_j'|^2 \right)^{1/2}$$

is bounded. The Fubini–Study derivative measures the length distortion by f from the Euclidean metric in $\mathbf{C}$ to the Fubini–Study metric in $\mathbf{P}^n$. Brody curves are named after R. Brody, who proved the following important fact: *If f is a non-constant holomorphic curve, then there exist $\rho_j > 0$ and $a_j \in \mathbf{C}$ such that $f(\rho_j z + a_j) \to g(z)$, uniformly on compact subsets of $\mathbf{C}$, where g is a non-constant Brody curve.* This fact has numerous applications, and it gives a motivation for the study of Brody curves. Another reason is the theory of mean dimension of Gromov and Lindenstrauss [14], see also [71], where the space of holomorphic curves with the property $\|f\| \leq 1$ plays an important role.

A trivial estimate shows that for Brody curves we have $T(r,f)=O(r^2)$, and this is best possible. However this growth estimate can be substantially improved if f omits several hyperplanes:

Theorem. [3, 21] *If f is a Brody curve omitting n hyperplanes in general position, then*

$$T(r,f)=O(r),\quad r\to\infty.$$

The case $n=1$ is due to Clunie and Hayman [10].

There is a remarkable conjecture, due to da Costa and Duval, which generalizes this theorem.

Conjecture. [12] *If f is a Brody curve whose image is not contained in the union of hyperplanes $L_1,\ldots,L_q$ in general position, then*

$$(q-n+1)T(r,f)\leq\sum_{j=1}^{q}N(r,L_j,f)+O(r).$$

This improves the constant in Cartan's Second Fundamental Theorem by 2 at the price of somewhat worse error term. If n values are omitted by f, we take $q=n$ and obtain the Theorem above. When $n=1$ one obtains

Corollary. [12] *If f is a meromorphic function with bounded spherical derivative, then*

$$T(r,f)=N(r,a,f)+O(r)\quad \textit{for every}\quad a\in\overline{\mathbf{C}}.$$

This implies the theorem of Hayman and Clunie, if the value a is omitted.

Using the methods of the PI, [21, 3], da Costa and Duval proved the Corollary, and obtained several other special cases of their Conjecture. They also proved the weak form of the Conjecture with the error term $o(r^2)$.

The PI plans to work on this Conjecture, trying to prove it in full generality. This will require a substantial improvement of the methods used in [21, 3, 12].

5. Inversion of the Second Fundamental Theorem

The complete statement of the Second Fundamental Theorem (SFT) of Cartan mentioned in the previous section is

$$\sum_{j=1}^{q}m(r,L_j,f)+N_1(r,f)\leq(n+1)T(r,f)+S(r,f),\tag{6}$$

where the $m(r, L_j, f)$ are the "proximity functions",

$$m(r, L, f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \frac{1}{[f(re^{it}), L]} dt,$$

and $[f(z), L]$ means the Fubini–Study distance from $f(z)$ to the hyperplane L, $N_1(r, f)$ is the Nevanlinna counting function of zeros of the Wronskian determinant $W(f_0, \ldots, f_n)$, and $S(r, f)$ is an error term which satisfies $S(r, f) = o(T(r, f))$ outside some small exceptional set of r.

When $n = 1$, this is the same as the SFT of Nevanlinna for a meromorphic function $f = f_1/f_0$, and L_j are just distinct points on $\mathbf{P}^1 = \overline{\mathbf{C}}$. Since the very beginning of Nevanlinna theory, the question was asked whether the asymptotic inequality expressed by the SFT can be reversed. Indeed, the SMT of Nevanlinna was always considered as a generalization of the Riemann–Hurwitz formula to infinite coverings, but the Riemann–Hurwitz formula is an equality, while the SMT is only an inequality.

There are several important but narrow classes of meromorphic functions for which the SFT of Nevanlinna is indeed an asymptotic equality, a survey of these results is given in [74], and little was added to the subject since 1955 until very recently, except the result of PI in [22] which proves the asymptotic equality in Nevanlinna's SMT under the assumption that $n = 1$ and $N_1(r, f) = o(T(r, f))$. One difficulty is that infinitely many hyperplanes L_j can make a substantial contribution to the left hand side of (6) as $r \to \infty$, but the infinite sum in the left hand side makes no sense for a fixed r. K. Yamanoi [75] proposed the following definitions for $n = 1$.

$$m_q(r, f) = \sup_{(a_1, \ldots, a_q) \in \overline{\mathbf{C}}^q} \frac{1}{2\pi} \int_{-\pi}^{\pi} \max_{1 \leq j \leq q} \log \frac{1}{[f(re^{it}), a_j]} dt, \tag{7}$$

With this definition, he proved that

$$m_q(r)(r, f) + N_1(r, f) = (2 + o(1))T(r, f) + S(r, f),$$

where $q(r) \to \infty$ sufficiently slowly.

The PI proposes to do the same in arbitrary dimension. The definition (7) can be extended literally, using the hyperplanes L_j instead of points a_j. Then the conjecture is that for linearly non-degenerate curves

$$m_{q(r)}(r, f) + N_1(r, f) = (n + 1)T(r, f) + S(r, f). \tag{8}$$

This problem is difficult: Yamanoi's method does not apply in higher dimensions. The PI begins with consideration of a very special case, when one can expect to obtain complete results, namely the holomorphic curves whose coordinates form a basis of solutions of a linear differential equation with polynomial coefficients [23].

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