On the number of solutions of some transcendental equations

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Abstract

We show that the number N of solutions of the equation

$$\log|z| = f(z),$$

where f is a rational function of degree d, satisfies $d \leq N \leq 5d$, and this is best possible.

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The following question was asked on Math Overflow [6]. Let p, q be polynomials of degrees m, n. How many solutions can the equation

$$p(z)\log|z| + q(z) = 0 \tag{1}$$

have?

Theorem. *The number N of solutions of equation (1), counting multiplicity, satisfies*

$$\max\{m, n\} \leq N \leq 3\max\{m, n\} + 2m.$$

Examples.

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1. First we show that the lower estimate is exact for all m, n. Let p, q be arbitrary polynomials of degrees $m \leq n$, without common roots, and consider the point $w = ir$ with $r > 0$ large enough (depending on p, q). Then the equation

$$p(z) \log |z| + q(z) + wp(z)$$

has $\max\{m, n\}$ solutions near poles of q/p: the number of solutions near a pole is the same as the multiplicity of this pole. Similarly, if $m > n$, the similar statement holds for about the equation

$$(p(z) + wq(z)) \log |z| + q(z) = 0,$$

solutions are near the poles of p/q. Thus the lower estimate in the Theorem is exact for all m, n.

2. $m = 0$. Taking $p(z) = 1$ and $q(z) = -cz$ we obtain 3 solutions when $c > 0$ is small enough. More generally, with $p(z) = 1$, $q(z) = 1 - z^n$, $n \geq 2$, we have $3n$ solutions. Indeed, there are two on the positive ray, $z_1 = 1$ and $z_2 \in (0, 1)$, and thus two on each ray $\arg z = 2\pi k/n$, $0 \leq k \leq n-1$, and one on each ray $\arg z = \pi(2k+1)/n$, $0 \leq k \leq n-1$. So the Theorem is exact when $m = 0$ and arbitrary n.

3. $n = 0$. For and arbitrary m, consider the equation

$$m \log |z| = -\frac{6 \log 2}{8z^m + 1}. \tag{2}$$

When $m = 1$, we have three positive roots, $1/16, 1/8, 1/4$, and two negative solutions which exist by the intermediate value theorem. Making the change of the variable $z \mapsto z^m$ in the equation with $m = 1$ we obtain equation (2) with $5m$ solutions. So the upper estimate in the Theorem is exact when $n = 0$ and m is arbitrary.

4. $n \leq m$. Previous example can be perturbed as follows. Choose a polynomial q of degree $n \leq m$ which is close to 1 on a compact set containing all $5m$ solutions of (2). As all solutions of (2) are non-degenerate, the inverse function theorem will guarantee that the number of solutions of

$$m \log |z| = -\frac{6 \log 2 q(z)}{8z^m + 1}$$

is at least $5m$ when q is sufficiently close to 1. This shows that the upper estimate in the Theorem is best possible for all $n \leq m$.

An explicit example with $m = n$ is

$$n \log |z| = 3 \log 2 \frac{z^n - 1}{z^n + 1}. \tag{3}$$

When $n = 1$ this equation has 5 real solutions: $1, 2, 1/2$ and two negative solutions, whose existence is evident from the intermediate value theorem. Making the change of the variable $z \mapsto z^n$ we obtain equation (9) with $5n$ solutions.

5. $n = 2m$. Take $c = 0.015$, and consider the equation

$$\log |z| = 3 \log 2 \left(1 - c(z-1)\right) \frac{z-1}{z+1}. \tag{4}$$

This is a small perturbation of (3) with $n = 1$, and one can easily check that it has 5 real solutions near those of (3) plus one negative solution ≈ -58.25 which exists by the intermediate value theorem. So the total number of real solutions is at least 6.

Let f be the right hand side of (3). Then f has two real critical points, $x_1 \approx 10.72$ and $x_2 \approx -12.72$ with critical values $y_1 \approx 2.93$ and $y_2 \approx 1.47$. This shows that there is a curve γ in the upper half-plane with endpoints x_1, x_2 on which f is real. As we have

$$\log |x_1| \approx 2.37 < y_1,$$

and

$$\log |x_2| \approx 2.53 > y_1,$$

we conclude that equation (4) must have a solution in the upper half-plane, and by symmetry, another one in the lower half-plane. This makes the total number of solutions 8.

Making the change of the variable $z \mapsto z^m$, we obtain an equation with $(m, n) = (m, 2m)$ having $8m = 3 \cdot 2m + 2m$ solutions. This shows that the upper estimate in the Theorem is exact when $n = 2m$.

6. $m = 3n$. Consider the equation

$$\log |z| = f(z), \tag{5}$$

with

$$f(z) = 3 \log 2 \left(1 - 0.015(z-1)\right) \left(1 + 0.0018(z-1)\right) \frac{z-1}{z+1},$$

which is a perturbation of the previous example. This equation has 7 real solutions,

$$(1.801, 1, 0.470, -0.083, -17.87, -150.1, -239.6).$$

First six are close to the solutions of (10).

Function f has three real critical points and a critical point at ∞, all simple. Two critical points are close to the critical points of the right hand side of (4), and additional one is approximately -241.79. Three critical points $x_1 < x_2 < x_3$ and corresponding critical values satisfy the inequalities:

$$x_1 < x_2 < x_3, \quad f(x_1) < \log|x_1|, \quad f(x_2) > \log|x_2|, \quad \log|x_3| > f(x_3). \quad (6)$$

These inequalities are verified with Maple, for example

$$x_1 \approx -241.7888, \quad f(x_1) \approx 5.479256 < 5.48806 \approx \log|x_1|.$$

The set in the upper half-plane, where f is real consists of two disjoint curves, γ_1 from x_1 to ∞ and γ_2 from x_2 to x_3. It follows from the inequalities (6) that our equation (5) has at least one solution on each of these curves. This makes the total number of solutions at least $7 + 2 + 2 = 11$, and shows that the upper estimate in the Theorem is exact when $(m, n) = (1, 3)$. We found the roots $-241.790963 + 13.653315i$ on γ_1 and $-8.53 + 10.28i$ on γ_2. Changing the variable $z \mapsto z^m$ in (5) gives an equation with $11m$ solutions, and $n = 3m$.

Proof of the Theorem.

We follow the method based on the combination of a topological degree computation with Fatou's theorem from holomorphic dynamics. The method was used for the first time in [5], see also [1] and a survey [4]. The difference of our argument in comparison with previous applications of the method is that we reduce (1) to an equation (10) with infinitely many solutions, but it is still possible to obtain the desired estimate.

Rewrite our equation (1) as

$$g(z) := \log|z|^2 + 2q(z)/p(z) =: \log|z|^2 + f(z) = 0 \quad (7)$$

Function g is a continuous map of the Riemann sphere into itself, $g(0) = g(\infty) = \infty$. We recall the definition of topological degree [7, Ch. II, §2]. Take a regular value w that is such w that for all solutions of the equation

$g(z) = w$ the map g is continuously differentiable near z and the Jacobian determinant $J_g(z) \neq 0$. Then the degree $\deg g$ is the sum over the full preimage of w of $\operatorname{sgn} J_g(z)$. Taking $w = ir$ with large real r we will find $(n-m)^+$ large preimages and m preimages near poles. Evidently $J_g(z) > 0$ at all these preimages. So the degree is $\deg g = \max\{n, m\}$. Let N^+, N^- be the numbers of solutions of $g(z) = 0$ where J_g is positive and negative. Then

$$N^+ - N^- = \max\{n, m\}. \tag{8}$$

The lower estimate $N = N^+ + N^- \geq \max\{m, n\}$ follows.

Computing the Jacobian we obtain

$$J_g = |g_z|^2 - |g_{\overline{z}}|^2 = \frac{1}{|z|^2}\left(|1 + zf'(z)|^2 - 1\right). \tag{9}$$

Now we transform equation (7) into the form

$$e^{-f(z)}/z - \overline{z} =: h(z) - \overline{z} = 0. \tag{10}$$

The set of solutions of (7) *is contained* in the set of solutions of (10); equation (10) can have infinitely many solutions, for example $\overline{z} = e^z/z$ has infinitely many solutions.

Computing the Jacobian of $h(z) - \overline{z}$ we obtain

$$|h'(z)|^2 - 1 = \frac{|e^{-2f(z)}|}{|z|^4}|1 + zf'(z)|^2 - 1. \tag{11}$$

If z is a solution of (7) that is $|z|^2 = \exp(-f(z))$, then the Jacobians (11) and (9) *have the same sign.*

Thus

$$N^- \leq n^-, \tag{12}$$

where n^- is the number of solutions of $h(z) - \overline{z} = 0$ with negative Jacobian. For these solutions we have $|h'(z)| < 1$, so they are exactly the *attracting fixed points* of the antiholomorphic function $\overline{h(z)}$.

Now we use the generalized Fatou's theorem [1] which says that the number of attracting fixed points of a holomorphic or of an antiholomorphic function does not exceed the number of singular values. The number of singular values of h is easy to estimate. From the explicit expression of h' (see (11)) the critical points are zeros of $1 + zf'$ in $\mathbf{C}$, so there is at most max$\{n+m, 2m\}$ of them. The asymptotic values of h can be only $0, \infty$, so they do not contribute. Thus

$$N^- \leq n^- \leq \max\{m+n, 2m\}.$$

Using (8), $N^+ = N^- + \max\{n, m\}$, we obtain that the total number $N^+ + N^-$ of solutions of (1) is at most $3n + 2m$ when $n > m$ and $5m$ otherwise. This proves the upper estimate.

Corollary. If f is a rational function of degree d, then the equation

$$\log |z| = f(z) \tag{13}$$

has at most $5d$ solutions, and this is best possible.

This can be compared with the theorem of Khavinson and Neumann [3] which can be stated as follows: *For a rational function f of degree d, the equation*

$$|z|^2 = f(z) \tag{14}$$

has at most $5d$ solutions. This is also best possible [8].

For the case when f is a polynomial, our theorem gives the upper estimate $3d$ and the result of a theorem of Khavinson and Swiatek gives the same estimate for equation (14), and this is also best possible [2].

The method permits to handle equations of the form $|z|^{2/k} = f(z)$, but is not clear how to extend these results to other non-analytic functions in the left hand side, for example, to the equation $|z|^3 = f(z)$.

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