1. Let $y(t)$ be the amount of the substance in the pool, in kilograms. Then

$$y'(t) = ab - ky - by/V,$$

Here the first summand reflects pumping into the pool, second term reflects the radioactive decay, and the third term is for overflow from the pool to the river. The constant k is related to the half-life by the formula $k = \log 2/T$ (see the book). This is a separable equation, but the problem only asks for the equilibrium. Equilibrium happens when

$$y = \frac{ab}{\log 2/T + b/V}.$$

This is the amount of the substance in the equilibrium. For concentration, one has to divide by V. We obtain

$$\frac{ab}{V \log 2/T + b}.$$

So by making large enough pool we can decrease the concentration to any given amount. Especially when the half-life is not too long.

2. Let $x(t)$ be the amount of the chemical in the tank A, and $y(t)$ in the tank B. We have

$$x' = -20x/100, \quad y(0) = 1,$$

nd

$$y' = 20x/100 - 20y/100, \quad y(0) = 0.$$

Solving the first equation we get $x(t) = e^{-t/5}$. Now the second one becomes a linear first order equation

$$y' = e^{-t/5}/5 - y/5.$$

Solving with the initial condition $y(0) = 0$, we obtain

$$y(t) = \frac{1}{5} t e^{-t/5}.$$

The maximum is at $t = 5$, and we have e^{-1} kg of the chemical at this moment.

3, 4 were solved in class.

5. The right hand side factors as $(x-2)^2(x+2)$, so 2 is unstable and -2 iss unstable but semistable.

6. This is homogeneous. Setting $x = wy$ (y is the independent variable) we get $w'y = \sqrt{w^2+1}$. This is separable

$$\frac{dw}{\sqrt{w^2+1}} = \frac{dy}{y},$$

$$\sinh^{-1} w = \log|y| + C.$$

$$w = \sinh(\log|y| + C), \quad x = y \sinh(\log|y| + C).$$

7. This is neither homogeneous nor exact but can be converted into a homogeneous one by introducing new variables u, v by the fomulas

$$y = u - 1/2, \quad x = v - 3/2.$$

so that Then for u and v we have a homogeneous equation

$$\frac{du}{dv} = -\frac{u+v}{v-u}.$$

Such equation we solved in class. After solving this we return to x, y by the above formulas.

8. This is exact.

$$(x+y+2)dx + (1+x-y)dy = 0 = Mdx + Ndy.$$

$M_y = 1 = N_x$. The implicit solution is

$$\frac{1}{2}x^2 - \frac{1}{2}y^2 + xy + 2x + 2y + C = 0.$$

9. This equation does not belong to any class we studied, and finding a general solution is difficult. However, it is only asked for a solution of the initial value problem, and this has an evident solution $y(x) \equiv= 0$. Existence and uniqueness theorem guarantees that there are no other solutions.

10.

$$-\frac{1}{2}t\cos t + c_1 \cos t + c_2 \sin t.$$