Practice problems for the final

You should be able to:
– describe the set of solutions of any system of linear equations
– determine whether a system of vectors is linearly independent
– find a span of any given system of vectors
– find a basis in any given subspace
– write the matrix of a linear transformation with respect to a given basis
– find the kernel and image of any linear transformation
– find a projection of any vector on any given subspace
– find the distance between a vector and a subspace
– construct an orthonormal basis in any given subspace
– evaluate any given determinant of moderate size
– find the inverse matrix by row operations and using determinants
– find eigenvalues and eigenspaces of any given linear transformation
– to determine whether a matrix is diagonalizable and if this is so, to diagonalize it.
– find a basis consisting of eigenvectors
– find the exponential of any given matrix
– solve linear difference equations with constant coefficients.

Below are some typical problems selected from the old exams.

1. Find the relation between det(exp(A)) and tr(A) for diagonalizable matrices A.

2. Let $P_2$ be the space of all polynomials of degree at most 2. Consider the linear transformation $T : P_2 \rightarrow P_2$ given by $T(f) = 2f' - f''$.
   a) Find the matrix of $T$ with respect to the basis $\{1, t, t^2\}$.
   b) Is this transformation diagonalizable?

3. Let

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$ 

   a) Find the image of $A$.
   b) Find an orthonormal basis in the image of $A$.
   c) Find the projection of $b$ onto the image of $A$.
   c) Find the distance from $b$ to the image of $A$.  

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4. Consider the following set of vectors in $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ -2 \\ a \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ b \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ c \end{bmatrix} \right\}.$$

a) What relation(s) should $a, b, c$ satisfy for $S$ to be linearly independent?

b) For which $a, b, c$ the set $S$ is orthogonal?

5. Consider the following system of equations

$$\begin{align*}
x + 2y + 6z &= 2 \\
y + 2mz &= 0 \\
mx + 2z &= 1
\end{align*}$$

For which values of $m$ does this system have

a) no solutions?
b) infinitely many solutions?
c) a unique solution?

6. Is it true that $\text{rank}(A) = \text{rank}(A^TA)$ for every matrix $A$, not necessarily square? (If yes, explain why, if no, give an example).

7. Solve the difference equation

$$a_{n+2} = 5a_{n+1} - 5a_n, \quad a_0 = 7, \quad a_1 = 17.$$ 

8. Is it true that every $m \times n$ matrix of rank one is a product of a $m \times 1$ matrix and a $1 \times n$ matrix, that is

$$A = \text{(column)} \times \text{(row)} \quad ?$$

(If yes, explain why, if no, give an example. If this looks too hard, consider the $2 \times 2$ case.)

9. Let $A$ be a $4 \times 4$ matrix, and $\det(A) = -2$. Find

$$\det(AA^T), \quad \det(A^3), \quad \det(-2A), \quad \det(A^{-1}).$$

10. Find the values of $a, b$ and $c$ such that

$$A = \begin{bmatrix} 3 & a & b \\ 0 & -4 & c \\ 0 & 0 & 3 \end{bmatrix}$$

is diagonalizable.