Spherical polygons and differential equations

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Abstract

This is an exposition of some results on classification of spherical polygons with prescribed interior angles and prescribed images of vertices under a conformal map onto the unit disk.

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This paper contains an exposition of the results from [5] and [6].

A polygon is a surface homeomorphic to the closed disk, with several marked points on the boundary called corners, equipped with a Riemannian metric of constant curvature $K$, such that the sides (arcs between the corners) are geodesic, and the metric has conical singularities at the corners.

A conical singularity is a point near which the length element of the metric is

$$ds = \frac{2\alpha|z|^{\alpha-1}|dz|}{1 + K|z|^2},$$

where $z$ is a local conformal coordinate. The number $2\pi\alpha > 0$ is the angle at the conical singularity. The interior angle of our polygon is $\pi\alpha$. These angles can be arbitrarily large. Every polygon can be mapped conformally onto the unit disk. We consider the problem of classification up to isometry of polygons with prescribed angles and prescribed corners.

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By *prescribed corners* we mean that the images of the corners on the unit circle are prescribed.

**Flat polygons,** $K = 0$. The angles must satisfy the condition
\[ \sum \alpha_j = n - 2. \]
For any given angles and prescribed corners, there exists a polygon, which is unique up to scaling.

Proof: Christoffel–Schwarz formula.

**Hyperbolic polygons,** $K < 0$. The angles must satisfy
\[ \sum \alpha_j < n - 2. \]
For any given angles and prescribed corners, there exists a unique polygon (E. Picard [13, 14, 15, 16], M. Troyanov [18]).

We study *spherical polygons*, assuming $K = 1$. The necessary condition on the angles,
\[ \sum \alpha_j > n - 2, \]
follows from the Gauss–Bonnet formula. If the angles are *sufficiently small*, for example $0 < \alpha_j < 1$, then we still have existence and uniqueness (M. Troyanov [18], F. Luo and G. Tian [10]).

**Spherical triangles** were classified by F. Klein [9], A. Eremenko [2], S. Fujimori, Y. Kawakami, M. Kokubu, W. Rossman, M. Umehara and K. Yamada [7].

If all $\alpha_j$ are not integers, the necessary and sufficient condition for the existence of a spherical triangle is
\[ \cos^2 \alpha_0 + \cos^2 \alpha_1 + \cos^2 \alpha_2 + 2 \cos \alpha_0 \cos \alpha_1 \cos \alpha_2 < 1, \]
and the triangle is unique.

If $\alpha_0$ is an integer, then the necessary and sufficient condition is that either $\alpha_1 + \alpha_2$ or $\alpha_1 - \alpha_2$ is an integer $m < \alpha_0$, with $m$ and $\alpha_0$ of opposite parity.

The triangle with an integer corner is not unique: there is a 1-parametric family when only one angle is integer, and a 2-parametric family when all angles are integer.
Developing map. A surface $D$ of constant curvature 1 is locally isometric to the standard sphere $S$. This isometry is conformal, has an analytic continuation to the whole polygon, and is called the developing map $f : D \to S$ [2], [1].

We say that spherical polygons are equivalent if their developing maps differ by a post-composition with an element of $PSL(2)$.

Let us choose the upper half-plane $H$ as the conformal model of our polygon, with $n$ corners $a_0, \ldots, a_{n-1}$, and choose $a_{n-1} = \infty$. Accordingly, we sometimes denote $a_{n-1}$ as $a_\infty$. The other corners are real numbers. Then $f : H \to S$ is a meromorphic function mapping the sides into great circles. By the Symmetry Principle, $f$ has an analytic continuation to a multi-valued function in $\bar{C} \setminus \{a_0, \ldots, a_{n-1}\}$ whose monodromy is a subgroup of $PSU(2) \sim SO(3)$.

Such a function must be a ratio of two linearly independent solutions of the Fuchsian differential equation

$$w'' + \sum_{k=0}^{n-2} \frac{1 - \alpha_k}{z - a_k} w' + \frac{P(z)}{\prod(z - a_k)} w = 0,$$

where $P$ is a real polynomial of degree $n - 3$ whose top coefficient can be expressed in terms of the $\alpha_j$. The remaining $n - 3$ coefficients of $P$ are called the accessory parameters. The monodromy group of this equation must be conjugate to a subgroup of $PSU(2)$. In the opposite direction, if a Fuchsian differential equation with real singularities and real coefficients has the monodromy group conjugate to a subgroup of $PSU(2)$, then the ratio of two linearly independent solutions restricted to $H$ is a developing map of a spherical polygon.

Thus classification of spherical polygons with given angles and corners is equivalent to the following problem:

For a Fuchsian equation with given real parameters $a_j, \alpha_j$, to find the real values of accessory parameters for which the monodromy group of that equation is conjugate to a subgroup of $PSU(2)$. These values of accessory parameters are in bijective correspondence with the equivalence classes of spherical polygons.

Spherical polygons with all integer angles. In this case, the developing map is a real rational function with real critical points. The multiplicities of the critical points are $\alpha_j - 1$. Such functions have been studied in great
The necessary and sufficient condition on the angles is
\[ \sum (\alpha_j - 1) = 2d - 2, \]
where \( d = \deg f \) is an integer, and \( \alpha_j \leq d \) for all \( j \). For given angles, there exist exactly \( K(\alpha_0 - 1, \ldots, \alpha_{n-1} - 1) \) of the equivalence classes of polygons, where \( K \) is the Kostka number: it is the number of ways to fill in a table with two rows of length \( d - 1 \) with \( \alpha_0 - 1 \) zeros, \( \alpha_1 - 1 \) ones, etc., so that the entries are non-decreasing in the rows and increasing in the columns.

**Polygons with two non-integer angles.** Let \( \alpha_0 \) and \( \alpha_{n-1} \) be non-integer, while the rest of the angles \( \alpha_j \) are integer.

Assuming \( \alpha_0 = 0 \) and \( \alpha_{n-1} = \infty \) we conclude that the developing map has the form
\[
f(z) = z^\alpha \frac{P(z)}{Q(z)},
\]
where \( \alpha \in (0,1) \) and \( P, Q \) are real polynomials. For this case, a necessary and sufficient condition on the angles is the following

**Theorem 1.** Let \( \sigma := \alpha_1 + \ldots + \alpha_{n-2} - n + 2 \).

a) If \( \sigma + [\alpha_0] + [\alpha_{n-1}] \) is even, then \( \alpha_0 - \alpha_{n-1} \) is an integer of the same parity as \( \sigma \), and \( |\alpha_0 - \alpha_{n-1}| \leq \sigma \).

b) If \( \sigma + [\alpha_0] + [\alpha_{n-1}] \) is odd, then \( \alpha_0 + \alpha_{n-1} \) is an integer of the same parity as \( \sigma \), and \( \alpha_0 + \alpha_{n-1} \leq \sigma \).

Finding all polygons with prescribed angles is equivalent in this case to solving the equation
\[
z(P'Q - PQ') + \alpha PQ = R
\]
with respect to real polynomials \( P \) and \( Q \) of degrees \( p \) and \( q \), respectively, where \( R \) is a given real polynomial of degree \( p + q \). The degree of the map
\[
W_\alpha : (P, Q) \mapsto z(P'Q - PQ') + \alpha PQ
\]
equals
\[
\left( \begin{array}{c} p + q \\ p \end{array} \right)
\]
(it is a linear projection of a Veronese variety), and one can show that when all roots of \( R \) are non-negative, all solutions \((P, Q) \in W_\alpha^{-1}(R)\) are real.
Enumeration of polygons with two adjacent non-integer angles. An important special case is when $a_0$ and $a_{n-1}$ are adjacent corners of the polygon, $2a_0$ and $2a_{n-1}$ are odd integers, while all other $\alpha_j$ are integers. Equivalence classes of such polygons are in bijective correspondence with odd real rational functions with all critical points real, given by

$$f(z) = g(\sqrt{z}),$$

where $f$ is the developing map of our polygon and $g$ is a rational function as above. By a deformation argument, this gives the following

**Theorem 2.** If the angles satisfy the necessary and sufficient condition given above, and the corners $a_0 = 0$ and $a_{n-1} = \infty$ are adjacent, then there are exactly

$$E(2a_0 - 1, \alpha_1 - 1, \ldots, \alpha_{n-2} - 1, 2a_{n-1} - 1)$$

equivalence classes of polygons, where $E(m_0, \ldots, m_{n-1})$ is the number of chord diagrams in $H$, symmetric with respect to $z \mapsto -\overline{z}$, with the vertices $0 = a_0 < a_1 < \ldots < a_{n-2} < a_{n-1} = \infty$ and $-a_1, \ldots, -a_{n-2}$, and $m_j$ chords ending at each vertex $a_j$.

If $a_0$ and $a_{n-1}$ are not adjacent, $E$ gives an upper bound on the number of equivalence classes of polygons.

One can express $E$ in terms of the Kostka numbers.

**Proposition.** Let $m_0$ and $m_{n-1}$ be even. Then

$$E(m_0, m_1, \ldots, m_{n-2}, m_{n-1}) = K(r, m_1, \ldots, m_{n-2}, s),$$

where positive integers $r$ and $s$ satisfy

$$r + s > m_1 + \ldots + m_{n-2}, \quad (1)$$

and can be defined as follows:

- If $\mu := (m_0 + m_{n-1})/2 + m_1 + \ldots + m_{n-2}$ is even, then $r = m_0/2 + k$, $s = m_{n-1}/2 + k$, where $k$ is large enough, so that (1) is satisfied.
- If $\mu$ is odd, then $r = (m_0 + m_{n-1})/2 + k + 1$, $s = k$, and $k$ is large enough, so that (1) is satisfied.

**Spherical quadrilaterals. Heun’s equation.** In the case $n = 4$ the Fuchsian equation for the developing map is the Heun’s equation

$$w'' + \left( \frac{1 - \alpha_0}{z} + \frac{1 - \alpha_1}{z - 1} + \frac{1 - \alpha_2}{z - a} \right) w' + \frac{A z - \lambda}{z(z - 1)(z - a)} w = 0,$$
where $A$ can be expressed in terms of $\alpha_j$, and $\lambda$ is the accessory parameter.

We can place three singularities at arbitrary points, so we choose $a_0 = 0$, $a_1 = 1$, $a_2 = a$, $a_3 = \infty$.

The condition that the monodromy belongs to $PSU(2)$ is equivalent to an equation of the form $F(a, \lambda) = 0$. This equation is algebraic if either one or two angles are integers.

Theorem 2 in the case of quadrilaterals with two integer and two non-integer angles specializes to the following

**Theorem 3.** The number of classes of quadrilaterals with two integer and two non-integer angles is at most

$$\min\{\alpha_1, \alpha_2, k + 1\},$$

where

$$k + 1 = \begin{cases} \frac{(\alpha_1 + \alpha_2 - |\alpha_0 - \alpha_3|)}{2} & \text{in case a)} \\ \frac{(\alpha_1 + \alpha_2 - \alpha_0 - \alpha_3)}{2} & \text{in case b).} \end{cases}$$

If $a > 0$ we have equality.

Here cases a) (when $\alpha_0 - \alpha_3$ is integer) and b) (when $\alpha_0 + \alpha_3$ is integer) are as in Theorem 1. Condition $a > 0$ means that the corners $a_1$ and $a_2$ with integer angles are adjacent.

**Quadrilaterals with non-adjacent integer angles.** Let $\delta = \max(0, \alpha_1 + \alpha_2 - [\alpha_0] - [\alpha_3])/2$.

**Theorem 4.** The number of equivalence classes of quadrilaterals with non-adjacent corners $a_1$ and $a_2$, with integer angles $\alpha_1$ and $\alpha_2$, is at least

$$\min\{\alpha_1, \alpha_2, k + 1\} - 2 \left[ \frac{1}{2} \min\{\alpha_1, \alpha_2, \delta\} \right],$$

where $k$ is the same as in Theorem 3.

Notice that in case b) of Theorems 1 and 3, the lower bound (2) becomes 0 when $\min\{\alpha_1, \alpha_2, k + 1\}$ is even and 1 if $\min\{\alpha_1, \alpha_2, k + 1\}$ is odd.

**Algebraic method.** In the case of quadrilaterals with one or two integer angles, our problem is equivalent to counting real solutions of an algebraic equation $F(a, \lambda) = 0$, expressing the fact that the Heun’s equation has $PSU(2)$ monodromy. Degree of this polynomial with respect to $\lambda$ gives the upper bound on the number of equivalence classes of quadrilaterals. The
polynomial $F$ is the spectral determinant of an eigenvalue problem for a certain finite Jacobi matrix. To see this, we re-write the Heun’s equation as an eigenvalue problem

$$z(z - 1)(z - a) \left( w'' + \left( \frac{1 - \alpha_0}{z} + \frac{1 - \alpha_2}{z - 1} + \frac{1 - \alpha_2}{z - a} \right) w' \right) + Azw = \lambda w.$$ 

The operator in the left-hand side can be represented by a Jacobi matrix acting on the sequence of the Taylor coefficients of $w$.

In the cases we consider, the problem can be reduced to the existence of a finite-dimensional eigenvector.

There is a natural quadratic form with respect to which the Jacobi matrix is symmetric [8]. This quadratic form is positive definite in the case when the corners with integer angles are adjacent, [12]. In the case when they are not adjacent we use Pontryagin’s theorem [17] on the matrices symmetric with respect to an indefinite form [11]. This method seems to work only in the cases when the eigenvalue problem is finite-dimensional.

**Geometric method.** The developing map is a local homeomorphism, except at the corners, of a closed disk $D$ to the standard sphere $S$. The sides are mapped to great circles. These great circles define a partition (cell decomposition) of the sphere. Taking the $f$-preimage of this partition, and adding vertices corresponding to the integer corners, we obtain a cell decomposition of $D$ which is called a net. Two nets are considered equivalent if they can be mapped to each other by an orientation-preserving homeomorphism of the disk, respecting labeling of the corners.

It is easy to see that a net, together with the partition of the sphere by the great circles, define the polygon completely.
Fig. 1. Partition of the Riemann sphere by two great circles.

Fig. 2. Nets with two adjacent integer corners.
Fig. 3. Chain of nets with two opposite integer corners.

Fig. 4. Partition of the Riemann sphere by four great circles (two views).
Fig. 5. Nets with four non-integer corners.
Our strategy is the following. First we classify all possible nets with given angles. Then we construct certain curves in the space of quadrilaterals with given angles, by moving the images of integer corners along the circles of the partition of the sphere $S$, and keeping the net fixed.

In the “good case” when the corners with integer angles are adjacent, we can show that the conformal modulus of the quadrilateral tends to 0 and $\infty$ on the ends of the curve. This proves the existence of a quadrilateral with prescribed angles and prescribed modulus. In the “difficult case” when the corners with integer angles are not adjacent, to construct the curves on which the modulus changes from 0 to $\infty$, one needs sometimes to paste together several curves with fixed nets.

The method is applicable, in principle, to all cases, no matter whether the accessory parameter problem is algebraic or not, but the computations become more complicated as the partition of the sphere by great circles contains more circles.

In the following pictures we choose the upper half-plane conformal model with corners $0, 1, a, \infty$, integer angles $\alpha_1$ and $\alpha_2$ at 0 and 1, non-integer angles $\alpha_0$ and $\alpha_3$ at $a$ and $\infty$, and plot the algebraic function $\lambda(a)$ which is defined by the condition that the monodromy of the Heun’s equation is unitary. The values $0 < a < 1$ correspond to quadrilaterals with opposite integer corners.
Fig. 6. $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_0 = \alpha_3 = 65/32$
Fig. 7. $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_0 = \alpha_3 = 255/128$
Fig. 8. $\alpha_1 = 6$, $\alpha_2 = 4$, $\alpha_0 = \alpha_3 = 5/4$
Fig. 9. $\alpha_1 = \alpha_2 = 3, \alpha_0 = \alpha_3 = \sqrt{2}$
Fig. 10. $\alpha_1 = \alpha_2 = 3$, $\alpha_0 = \alpha_3 = 15/8$
Fig. 11. $\alpha_1 = \alpha_2 = 3$, $\alpha_0 = \alpha_3 = 63/32$
References

[1] Q. Chen, Y. Wu and B. Xu, Developing map and character 1-form of a singular conformal metric of constant curvature one on a compact surface, arxiv:1302.6457.


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