

**MATH 519 Preparation Problems for the Qualifying exam,  
August 2003**

1. Stephan Banach used to carry two match boxes, one in each pocket. Each box initially contained  $n$  matches. Whenever he lighted a cigarette a box was chosen randomly with probability  $1/2$ . When he found that one box is empty, what was the expected number of matches in another box?
2. A marksman  $A$  hits a target under certain conditions with probability 0.6, marksman  $B$  with probability 0.5 and marksman  $C$  with probability 0.6. Once all three of them fired a salvo (each made one shot, simultaneously) and two bullets hit the target. What is the probability that one of these bullets was shot by  $C$ ?
3. A rod of length 1 is broken into two pieces, and then the larger piece is broken into two pieces. Each time the breaking point is chosen uniformly with respect to the length. What is the probability that one can make a triangle of the resulting three pieces?
4.  $n$  numbers are chosen independently and uniformly distributed on  $[1, 2]$ . Find the distribution of the natural logarithm of the smallest number.
5. From an urn that initially contained  $m \geq 3$  white balls and  $n$  black balls, one ball (of unknown color) was lost. To determine what was the color of the lost ball, they drew two balls randomly, and it turned out that these two balls were white. What is the probability that the lost ball was white?
6. Let  $X_1, \dots, X_n, \dots$  be a sequence of independent random variables taking the values 0 and 1 with probability  $1/2$ . Find the distribution of the random variable

$$S = \sum_1^{\infty} 2^{-n} X_n.$$

7. In XVI century, in one German state the following procedure was used to determine the length of one foot. On a Sunday, 16 men were taken randomly, their left feet were measured, added and divided by 16. The result was declared the “legal and correct foot”.

Given that the average length of a foot of an adult man is 262.5 mm and the standard deviation is 12 mm, answer the following questions:

a) What is the probability that two “legal and correct feet” determined from two independent groups will differ by more than 5 mm?

b) How many men should be used to ensure that the average length of their feet differs from 262.5 mm by at most 0.5 mm with probability .99.

8. An exterminator company claims that its treatment will kill a given flea with probability 99.9% and a flea egg with probability 99%. If there are 100 fleas and 100 flea eggs in the house, what is the probability that the treatment will solve the flea problem for this house? (“To solve the problem” means “to destroy all fleas and all flea eggs”)

9. a) Show that

$$f(x) = \frac{1}{\pi} \frac{c}{c^2 + x^2}, \quad c > 0$$

is a probability density. (The corresponding distribution is called the Cauchy distribution with parameter  $c$ ).

b) Find the characteristic function  $\phi$  of Cauchy distribution. (Hint: Find the characteristic function of an exponential distribution first, this may help).

c) What is the distribution of  $(X_1 + \dots + X_n)/n$ , where  $X_j$  are independent and have Cauchy distribution with the same parameter  $c$ ?