Rational maps with real multipliers

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Abstract

Let f be a rational function such that the multipliers of all repelling periodic points are real. We prove that the Julia set of such a function belongs to a circle. Combining this with a result of Fatou we conclude that whenever $J(f)$ belongs to a smooth curve, it also belongs to a circle. Then we discuss rational functions whose Julia sets belong to a circle.

MSC classes: 37F10, 30D05.

A simple argument of Fatou [4, Section 46] shows that if the Julia set of a rational function is a smooth curve then all periodic orbits on the Julia set have real multipliers, see also [8, Cor. 8.11]. This argument gives the same conclusion if one only assumes that the Julia set is *contained* in a smooth curve. By a smooth curve we mean a curve that has a tangent at every point.

We prove the converse statement:

Theorem 1. *Let $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map such that the multiplier of each repelling periodic orbit is real. Then either the Julia set $J(f)$ is contained in a circle or f is a Lattès map.*

Corollary 1. *If the Julia set of a rational function is contained in a smooth curve then it is contained in a circle.*

In fact, Theorem 1 holds if all repelling periodic points on some relatively open subset of $J(f)$ have real multipliers. It follows that even if a relatively

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open set of the Julia set is contained in a smooth curve, then the Julia set is contained in a circle.

This corollary generalizes the result of Fatou [4, Section 43] that whenever the Julia set of a rational function is a smooth curve, this curve has to be a circle or an arc of a circle. For another proof of the corollary, independent of Theorem 1, see [1]. We give a more precise description of the maps which can occur:

Theorem 2. *Let $f\colon \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ be a rational map whose Julia set $J(f)$ is contained in a circle C. Then there are the following possibilities:*
(i) C is completely invariant, in which case f^2 is a Blaschke product (that is both components of the complement of C are invariant under f^2). The Julia set is either C of a Cantor subset of C.

If (i) does not hold, then there is a critical point on C, and a fixed point $x_0 \in C$ whose multiplier satisfies $\lambda \in [-1, 1]$. Let I be the smallest closed arc on C which contains $J(f)$ and whose interior does not contain x_0. Then one of the following holds:
(ii) I is a proper arc which is completely invariant, and $J(f) = I$.
(iii) $f(I)$ strictly contains I. The Julia set is a Cantor subset of I in this case.
In Cases (ii) and (iii), for each critical point x of f in I there exists $N \geq 1$ such that $f^N(x) \notin$ interior(I) (where in Case (ii) $N = 1$). All critical points on $J(f)$ are pre-periodic.

Remarks. If f is a Blaschke product preserving a circle C, then $f : C \to C$ is a covering, so all three cases (i), (ii) and (iii) are disjoint. Chebyshev polynomials belong to the case (ii), and every polynomial satisfying (ii) is conjugate to $\pm T$, where T is a Chebyshev polynomial. If f in Case (iii) is a polynomial one can always take $N = 1$, but there are rational functions satisfying (iii) for which $N > 1$, see Example 3 at the end of the paper.

There are functions f satisfying (ii) which are not conjugate to polynomials. A parametric description of functions satisfying (ii) can be obtained using [3, Section 25][1].

Each f satisfying (ii) is conjugate to $B^2(\sqrt{z})$, where B is an odd rational function which leaves both upper and lower half-planes invariant, and whose Julia set equals $\bar{\mathbb{R}}$. In the opposite direction, if B is an odd rational function

[1]We use this opportunity to notice that the statement of these results of Fatou in the survey [2] is wrong. See Fatou's original paper for the correct statements.

which leaves both upper and lower half-planes invariant, and whose Julia set equals $\bar{\mathbb{R}}$ then $B^2(\sqrt{z})$ is a rational function whose Julia set is the ray $[0, \infty]$, and this function satisfies (ii).

In Cases (ii) and (iii) the interval I is equal to $C \setminus B_0$ where B_0 denotes the immediate basin of x_0 for the restriction of f on C. In Case (iii) there exist finitely many closed arcs on C such that the full preimage of their union is contained in this union. To prove this claim, take (within C) the preimages up to order N of B_0, where N is as in Theorem 2 and therefore the union K of the closures of these intervals contains all critical values in I. Hence $I \setminus K$ has the following properties: $I \setminus K$ contains the Julia set. As K is forward invariant, $\bar{\mathbb{C}} \setminus K$ is backward invariant. As every point of $J(f)$ (and therefore every point of $I \setminus K$ has all preimages in the closure of $I \setminus K$, and there are no critical values in the interior of $I \setminus K$, we conclude that the closure of $I \setminus K$ is backward invariant.

Theorem 1 is proved in Section 1. In Section 2, we prove Theorem 2 and discuss rational functions satisfying (iii) of Theorem 2.

1 Proof of Theorem 1

There are only finitely many repelling cycles which belong to the forward orbits of critical points. So there exists a repelling periodic point p of f of period N which does not lie on the forward orbit of a critical point. Replacing f by f^N we may assume that $N = 1$.

Let $\Psi\colon \mathbb{C} \to \bar{\mathbb{C}}$ be a holomorphic map which globally linearizes f at p, i.e.,

$$\Psi\lambda = f\Psi, \quad \Psi(0) = p, \quad D\Psi(0) \neq 0.$$

Here λ is the multiplier of p, so $\lambda := Df(p)$ is real. Such a map Ψ, which is also called a Poincaré function [11], always exists. It is uniquely defined by the value $D\Psi(0) \in \mathbb{C} \setminus \{0\}$ which can be prescribed arbitrarily.

Lemma 1. *If $z \in \Psi^{-1}(p)$ and p is not an iterate of a critical point then $D\Psi(z) \neq 0$.*

Proof. Since $\Psi(z) = f^n \Psi \lambda^{-n} z$, and Ψ is univalent in a neighborhood of 0, the result follows. □

We will use several times the following result of Ritt [10]:

Lemma 2. *The Poincaré function is periodic if and only if f is a Lattès map, or conjugate to $\pm T_n$, where T_n is a Chebyshev polynomial, or conjugate to $z^{\pm d}$.*

Rational functions with periodic Poincaré functions described in Lemma 2 will be called *exceptional*.

1.1 The linearizing map restricted to certain lines

A simple curve $\gamma : (0,1) \to \bar{\mathbb{C}}$ passing through a repelling periodic periodic point p of period N is called an *unstable manifold* for p if there exists a subarc γ_* of γ containing p so that f^N maps γ_* diffeomorphically onto γ. Similarly, we say that γ is an *invariant curve for* $p_{-m} \in f^{-m}(p)$ if γ is contained in an unstable manifold for p, $p_{-m} \in \gamma$ and $f^m(V_m \cap \gamma) \subset \gamma$ for some neighborhood V_m of p_{-m}.

Choose $Q \in \Psi^{-1}(p)\backslash\{0\}$. By Lemma 1, $D\Psi(Q) \neq 0$, so there exist small topological discs $\mathcal{O}_0 \ni 0$ and $\mathcal{O}_1 \ni Q$ so that $\Psi|\mathcal{O}_0$ and $\Psi|\mathcal{O}_1$ are univalent and $\Psi(\mathcal{O}_0) = \Psi(\mathcal{O}_1)$. Hence there exists a biholomorphic map $\mathcal{T} \colon \mathcal{O}_0 \to \mathcal{O}_1$ for which

$$\mathcal{T}(0) = Q \text{ and } \Psi \circ \mathcal{T} = \Psi \text{ restricted to } \mathcal{O}_0.$$

For convenience we may choose $\mathcal{O}_i$ so that $\lambda^{-1}\mathcal{O}_0 \subset \mathcal{O}_0$, for example we can take a round disc for $\mathcal{O}_0$.

First we show that the linearizing map Ψ is special on certain lines. To prove this, we shall use the following

Lemma 3. *For each $Q \in \Psi^{-1}(p)\backslash\{0\}$, there exists a sequence $z_n \to 0$ so that $\lambda^n z_n \to Q$ and $\Psi(z_n)$ is a repelling periodic point of period n. There exists a neighborhood $V_n \subset \mathcal{O}_0$ of z_n so that $f^n \colon \Psi(V_n) \to \Psi(\mathcal{O}_0)$ is biholomorphic.*

Proof. Take n so large that the closure of $\lambda^{-n}\mathcal{O}_1$ is contained in $\mathcal{O}_0$. Notice that

$$f^n|\Psi(\lambda^{-n}\mathcal{O}_1) = (\Psi|\mathcal{O}_1) \circ \lambda^n \circ (\Psi|\lambda^{-n}\mathcal{O}_1)^{-1},$$

and therefore $f^n|\Psi(\lambda^{-n}\mathcal{O}_1)$ is univalent on $V_n := \lambda^{-n}\mathcal{O}_1$. Moreover,

$$f^n(\Psi(\lambda^{-n}\mathcal{O}_1)) = \Psi(\mathcal{O}_1) = \Psi(\mathcal{O}_0),$$

and thus there exists $z_n \in V_n$ so that $\Psi(z_n)$ is a fixed point of f^n. ∎

Lemma 4. *Suppose that f is not an exceptional function from Lemma 2. Let p, Q and $\mathcal{T}\colon \mathcal{O}_0 \to \mathcal{O}_1$ be as above and let L be the line through 0 and Q. Let γ be the arc $\Psi(L \cap \mathcal{O}_0)$. Then*

1. *γ is an unstable manifold for the fixed point p;*
2. *there exists a sequence $z_n \in L$ so that $\Psi(z_n)$ is a periodic point of period n and γ is an unstable manifold for $\Psi(z_n)$;*
3. *for each n large enough, γ is an invariant curve for $\Psi(\lambda^{-n}Q)$ (which is in the backward orbit of p);*
4. *$\Psi(\mathcal{O}_0 \cap L) = \Psi(\mathcal{O}_1 \cap L)$ and so $\mathcal{T}$ maps $\mathcal{O}_0 \cap L$ diffeomorphically onto $\mathcal{O}_1 \cap L$;*
5. *the set $\{z \in \mathcal{O}_0 : D\mathcal{T}(z) \in \mathbb{R}\}$ is a finite union of real analytic curves, one of which is $\mathcal{O}_0 \cap L$.*

Proof. First we prove that

$$D\mathcal{T}(z_n) \in \mathbb{R}, \tag{1}$$

for the points z_n from Lemma 3. Let $x_n = \Psi(z_n)$ be the corresponding periodic points of period n. Then $\Psi\lambda^n = f^n\Psi$ implies

$$D\Psi(\lambda^n z_n)\lambda^n = Df^n(x_n)D\Psi(z_n).$$

Since $Df^n(x_n)$ and λ are real, it follows that $D\Psi(\lambda^n z_n)/D\Psi(z_n) \in \mathbb{R}$. We note that $\Psi(\lambda^n z_n) = f^n(\Psi(z_n)) = f^n(x_n) = x_n = \Psi(z_n)$ and $z_n \to 0$ and $\lambda^n z_n \to Q$ and therefore

$$\mathcal{T} z_n = \lambda^n z_n. \tag{2}$$

Hence $D\Psi(\mathcal{T} z_n)/D\Psi(z_n) \in \mathbb{R}$. This implies (1).

If $D\mathcal{T}$ is constant on $\mathcal{O}_0$ then $\mathcal{T}$ is an affine map, $\mathcal{T}(z) = az+Q$ where $a = D\mathcal{T}(0) \in \mathbb{R}\backslash\{0\}$. The identity $\Psi \circ \mathcal{T} = \Psi$ with a non-constant meromorphic function Ψ implies that $a = \pm 1$ and we conclude that Ψ is periodic. Then f is an exceptional function from Lemma 2, contrary to our assumption.

From now on we assume that $D\mathcal{T}$ is not constant. Then the set $X = \{z \in \mathcal{O}_0 : D\mathcal{T} \in \mathbb{R}\}$ is a finite union of real analytic curves. We are going to prove that $L \cap \mathcal{O}_0$ is one of these curves.

Without loss of generality we may assume that L is the real line.

Let β be a curve in X that contains infinitely many points z_n. As $\lambda^n z_n \to Q$, we conclude $\arg z_n \to 0$, so β is tangent to L at 0. Let $K > 0$ be the order of contact of β and L at 0. Since $D\mathcal{T}(0) \in \mathbb{R}\backslash\{0\}$, the curves $\mathcal{T}\beta$ and L have the same order of contact at Q. Since $z_n \in \beta$ is of the form $t_n + i(\tau t_n^K + o(t_n^K))$, whereas $\mathcal{T} z_n \in \mathcal{T}\beta$ is of the form $Q + at_n + o(t_n) + i(a\tau t_n^K + o(t_n^K))$, where $\tau \neq 0$, $t_n \in \mathbb{R}$, $a = D\mathcal{T}(0) \in \mathbb{R}\backslash\{0\}$ and $Q \in \mathbb{R}\backslash\{0\}$ (remember that we assumed $L = \mathbb{R}$). We have $\arg z_n = \arg \mathcal{T} z_n$ in view of (2), so we obtain for large n:

$$(a\tau t_n^K + o(t_n^K))/(Q + at_n + o(t_n)) = (\tau t_n^K + o(t_n^K))/t_n$$

Since $t_n \to 0$, this is only possible if $\tau = 0$. It follows that $z_n \in L$ for all large n.

As β and L intersect at infinitely many points z_n, we conclude that $\beta = L \cap \mathcal{O}_0$, and this proves property 5 of the lemma.

Now $\mathcal{T} : \mathcal{O}_0 \to \mathcal{O}_1$ is biholomorphic, $\mathcal{T}(0) = Q$ and $D\mathcal{T}(z)$ is real for real z. This implies property 4.

We put $\gamma = \Psi(L \cap \mathcal{O}_0)$. Then property 1 is evident: take $\gamma_* = \Psi(\lambda^{-1}(L \cap \mathcal{O}_0))$. Property 2 follows from $\lambda^n(V_n \cap L) = \mathcal{O}_1 \cap L$ (notation from Lemma 3), and the fact that $f^n : \Psi(V_n) \to \Psi(\mathcal{O}_0)$ is biholomorphic. This also implies property 3 because $\lambda^{-n} Q \in V_n$. ☐

1.2 The case that $\Psi^{-1}(J(f))$ is not contained in a line

We will use following notation: if γ is a curve through x then $T_x\gamma$ will denote its tangent line at x.

Lemma 5. *Assume that $\Psi^{-1}(J(f)) \not\subset L$. Then Ψ is a periodic function, and f is a Lattès map.*

Proof. Throughout the proof, Q, Ψ and $\mathcal{T}\colon \mathcal{O}_0 \to \mathcal{O}_1$ will be as defined before Lemma 3 (with $\mathcal{O}_1$ a neighborhood of Q).

Since $\Psi^{-1}(J(f)) \not\subset L$, there exists $Q^1 \in \mathbb{C} \setminus L$ so that $\Psi(Q^1)$ is in the backward orbit of p, say $f^m(\Psi(Q^1)) = p$. Define $Q' = \lambda^m Q^1$, then

$$\Psi(Q') = \Psi(\lambda^m Q^1) = f^m \Psi(Q^1) = p,$$

and thus Lemma 4 applies to the line L' through 0 and Q'.

As we assume that $\Psi^{-1}(J(f)) \notin L$, there is an infinite set of lines L' as above. Indeed, it is easy to see that whenever $\Psi^{-1}(J(f))$ is contained in a finite union of lines then it is actually contained in one line. We are going to prove that $D\mathcal{T}(z) \in \mathbb{R}$ for each such line L', thus concluding from part 5 of Lemma 4 that f is one of the functions listed in Lemma 2. For all those functions, except Lattès maps, $\Psi(J(f))$ is a line, so we will conclude that f is a Lattès map.

We denote by $\mathcal{O}'_0$ and $\mathcal{O}'_1$ neighborhoods of 0 and Q', respectively, such that Ψ is univalent in these neighborhoods and $\Psi(\mathcal{O}'_0) = \Psi(O'_1)$. We choose a round disc as $\mathcal{O}'_0$.

Applying Lemmas 3 and 4 to the point Q' we obtain a sequence $z_k \in L'$, $z_k \to 0$ such that $\lambda^k z_k \to Q'$ and $x_k = \Psi(z_k)$ are repelling periodic points of period k. Fix such a point $z = z_k \in \mathcal{O}'_0$, so that $\lambda^k z \in \mathcal{O}'_1$, and $x = \Psi(z)$ is the corresponding periodic point (of period k), which does not belong to the forward orbit of a critical point.

By statements 1, 2 and 4 of Lemma 4, $\gamma' := \Psi(L' \cap \mathcal{O}'_0) = \Psi(L' \cap \mathcal{O}'_1)$ is an unstable manifold for p and also an unstable manifold for x. Since $\Psi|\mathcal{O}'_0$ is univalent, the curve γ' is smooth and has no self-intersections. (We chose $\mathcal{O}'_0$ to be a disc, and so γ' is connected.) Since γ' is an unstable manifold of x, there exists a curve $\gamma'_* \subset \gamma'$ through x so that f^k maps γ'_* diffeomorphically onto γ'. That is, there exists a nested sequence of curves $\gamma'_{i,*} \supset \gamma'_{i+1,*} \ni p$ shrinking in diameter to 0 (with $\gamma'_{0,*} = \gamma'$) so that f^k maps $\gamma'_{i+1,*}$ diffeomorphically onto $\gamma'_{i,*}$.

Now also consider the linearization $\hat{\Psi}$ of f^k associated to the periodic point x, i.e.

$$f^k \hat{\Psi} = \hat{\Psi}\mu \quad \text{where} \quad \mu = Df^k(x). \tag{3}$$

Let $\gamma_{i,*}$ be the arcs defined a few lines above, and take i so large that there exists a curve $\hat{L}'_i$ containing 0 which is mapped by $\hat{\Psi}$ diffeomorphically onto $\gamma_{i,*}$. Note that $f^{ik}\colon \gamma_{i,*} \to \gamma'$ can be written as $\hat{\Psi} \circ \mu^i \circ (\hat{\Psi}|\hat{L}'_i)^{-1}$ and, since this map is a diffeomorphism onto, it follows that $\hat{\Psi}$ is also a diffeomorphism restricted to the curve $\hat{L}' := \mu^i \hat{L}'_i$, and that $\hat{\Psi}(\hat{L}') = \gamma'$. In particular there exists $\hat{w} \in \hat{L}'$ so that $\hat{\Psi}(\hat{w}) = p$.

Since γ' is an unstable manifold for x, the curve $\hat{L}'$ is invariant under $z \mapsto \mu z$. As the only smooth curve through 0 which is invariant under real multiplication is a line, $\hat{L}'$ must be contained in a line $\hat{M}$ through 0.

Let $z' = \mathcal{T}(z) \in \mathcal{O}_1$. For $j \geq 0$ large, $w_j := \lambda^{-jk}(z')$ is contained in $\mathcal{O}_0$. Note that $\Psi(w_j)$ tends to $p = \hat{\Psi}(\hat{w})$ as $j \to \infty$. Since $\hat{\Psi}$ is a diffeomorphism restricted to $\hat{L}'$, and $\hat{w} \in \hat{L}'$, then for j large enough there exist unique $\hat{w}_j$ such that $\hat{\Psi}(\hat{w}_j) = \Psi(w_j)$.

Note that

$$\hat{\Psi}(\mu^j \hat{w}_j) = f^{jk}\hat{\Psi}(\hat{w}_j) = f^{jk}\Psi(w_j) = \Psi(\lambda^{jk}w_j) = \Psi(z') = \Psi(z) = x.$$

Let $\hat{M}_j'$ be the line through 0 and $\mu^j \hat{w}_j$ and let $\hat{M}_j \subset \hat{M}_j'$ be an open line segment containing the line segment $[\hat{w}_j, 0]$ and contained in a small neighborhood of $[\hat{w}_j, 0]$. By Lemma 1, there exist neighborhoods $\hat{\mathcal{O}}_0 \ni 0$ and $\hat{\mathcal{O}}_1 \ni \mu^j \hat{w}_j$ on each of which $\hat{\Psi}$ is biholomorphic, and $\hat{\Psi}(\hat{\mathcal{O}}_0) = \hat{\Psi}(\hat{\mathcal{O}}_1)$. Next apply Lemma 4 to the map $\hat{\Psi}$ (taking instead of L, Q the line $\hat{M}_j'$ and $\mu^j \hat{w}_j \in \hat{\Psi}^{-1}(x)$). This gives that $\hat{\Psi}(\hat{M}_j' \cap \hat{\mathcal{O}}_0)$ is an invariant manifold for x and that $\hat{\Psi}(\hat{M}_j' \cap \hat{\mathcal{O}}_0) = \hat{\Psi}(\hat{M}_j' \cap \hat{\mathcal{O}}_1)$. By statements 3 and 4 of Lemma 4, there exist small neighborhoods $\hat{V}_j$ of $\hat{w}_j$, $\hat{V}_j^1$ of $\mu^j \hat{w}_j$ and $\hat{V}_j^0$ of 0 so that

$$f^{jk}(\hat{\Psi}(\hat{M}_j' \cap \hat{V}_j)) = \hat{\Psi}(\hat{M}_j' \cap \hat{V}_j^1) = \hat{\Psi}(\hat{M}_j' \cap \hat{V}_j^0) \subset \hat{\Psi}(\hat{M}_j). \tag{4}$$

The first equality holds in view of (3) since μ is real. Since $\hat{w}_j$ lies close to $\hat{w}$ and $\hat{\Psi}$ is a diffeomorphism restricted to $[\hat{w}, 0]$, $\hat{\Psi}(\hat{M}_j)$ is a smooth curve which lies close to $\hat{\Psi}([\hat{w}, 0])$ (which is the subarc of γ' connecting p and x defined by $\Psi([0, z])$). It follows that there exists a curve $M_j \subset \mathcal{O}_0$ through w_j and z so that $\Psi(M_j) = \hat{\Psi}(\hat{M}_j)$. By (4), there exists a small neighborhood V_j of w_j so that

$$\Psi(\lambda^{jk}(M_j \cap V_j)) = f^{jk}(\Psi(M_j \cap V_j)) \subset \Psi(M_j) = \Psi(\mathcal{T}M_j).$$

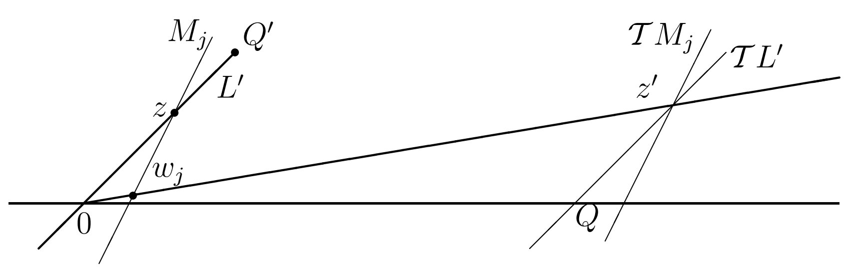

Figure 1: The curves used in the proof of Lemma 5. The thinly drawn curves are not necessarily line-segments.

Since $\lambda^{jk}w_j = z'$, the curves $\lambda^{jk}M_j$ and $\mathcal{T}M_j$ both go through z' and by the previous inclusion these curves agree near z'. In particular, the tangents of these curves at z' agree:

$$T_{z'}(\lambda^{jk} M_j) = T_{z'}(\mathcal{T} M_j). \tag{5}$$

The left hand side of (5) is equal to $T_{w_j} M_j$. Note that $\hat{M}_j$ converges to $\hat{M}$ as $j \to \infty$, that $\hat{\Psi}(\hat{L}') = \gamma' = \Psi(L' \cap \mathcal{O}_0')$ with $\hat{L}' \subset \hat{M}$, and $\Psi(M_j) = \hat{\Psi}(\hat{M}_j)$. Hence M_j converges in the C^1 sense to a segment in L'. (Here we use that $\hat{\Psi}$ is a diffeomorphism on a neighborhood of $[0, \hat{w}]$).

We get therefore that $T_{z'}(\lambda^{jk} M_j) \to T_0 L' = T_z L'$ and that the right hand side of (5) converges to $T_{z'}(\mathcal{T} L')$. Combined, it follows that

$$T_z(L') = T_{z'}(\mathcal{T} L')$$

and so $D\mathcal{T}(z) \in \mathbb{R}$. Since this holds for a whole sequence of points $z = z_k \in L'$ we obtain that $D\mathcal{T}(z) \in \mathbb{R}$ for all $z \in L'$.

As there are infinitely many such lines L', this implies that $D\mathcal{T}$ is constant, thus f is a Lattès map. □

1.3 Completion of the proof of Theorem 1

If f is not a Lattès map, Lemma 5 implies that $\Psi^{-1}(J(f)) \subset L$. Without loss of generality we may assume that that L is the real line.

We recall that the order ρ of a meromorphic function Ψ is defined by the formula

$$\rho = \limsup_{r \to \infty} \frac{\log T(r, f)}{\log r},$$

where $T(t, f)$ is the Nevanlinna characteristic [9]. According to a theorem of Valiron, [11, §51] the order of a Poincaré function Ψ satisfying the equation

$$\Psi \lambda = f^N \Psi$$

can be found by the formula

$$\rho = N \log \deg f / \log |\lambda|. \tag{6}$$

We claim that under the assumption that $J \subset \Psi(L)$, one can always find infinitely many periodic points p such that the orders of the corresponding functions Ψ will satisfy $\rho \leq 1 + \epsilon$, for any given $\epsilon > 0$.

To prove the claim, we consider the measure of maximal entropy μ and the characteristic exponent

$$\chi(z) = \lim_{n\to\infty} \frac{1}{n} \log |(Df^n)(z)|.$$

The reader may consult the survey [2] about these notions. According to the multiplicative ergodic theorem, this limit exists a. e. with respect to μ, and it is equal a. e. to the average characteristic exponent

$$\chi := \int \log |Df(z)| d\mu(z). \tag{7}$$

The average characteristic exponent is related to the Hausdorff dimension $HD(\mu)$ of measure μ by the formula

$$\chi = \frac{\log \deg f}{HD(\mu)},$$

proved in [7]. As μ is supported on the Julia set, and the Julia set is the image of a line under a meromorphic function, we conclude that $HD(\mu) \leq 1$. So

$$\chi \geq \log \deg f. \tag{8}$$

Now, μ is a weak limit of atomic probability measures μ_N equidistributed over periodic points of period N. Then (7) and (8) imply that there is infinitely many periodic points p of periods N such that the multipliers λ of these points satisfy $\log |\lambda| \geq (1-\epsilon)N(\log \deg f)$. We conclude from Valiron's formula (6) that the order of the Poincaré function Ψ is at most $(1-\epsilon)^{-1}$, as advertised.

We may assume without loss of generality that $\{0, \infty\} \subset J(f)$ and that $p = \Psi(0) \in \mathbb{R}$. (This can be achieved by conjugating f by a fractional-linear transformation). As we also assume that $L = \mathbb{R}$, the zeros a_j and poles b_j of Ψ are all real. Taking $\epsilon = 1/3$ we obtain a Poincaré function of order at most 3/2. According to a theorem of Nevanlinna [5, 6, 9], our function Ψ of order less than 2 has a canonical representation

$$\Psi(z) = be^{az} \frac{\prod_j (1 - z/a_j) e^{z/a_j}}{\prod_j (1 - z/b_j) e^{z/b_j}}$$

As $b = \Psi(0)$, a_j and b_j are all real, we conclude

$$\Psi(z) = e^{icz} g(z), \tag{9}$$

where the function g is real on the real line, and the constant $c = \Im a$ is real. If $c = 0$ then $\Psi(\mathbb{R})$ is contained in the real line and this completes the proof.

Suppose now that $c \neq 0$. We assume as before that the point p does not belong to the critical orbit of f. Then p is not a critical value of Ψ. Suppose that $\Psi(z_n) = p$ for $n = 1, 2, \ldots$, then all z_k are real. Put $\Psi_n(z) = \Psi(z - z_n)$. Let U be a small interval around zero on the real line such that Ψ is univalent in U. Let $\gamma_n = \Psi_n(U)$. These are analytic curves, and (since the Julia set is perfect) any two of them have infinitely many intersection points having an accumulation at $p = \Psi(0)$. We conclude that all these γ_n are reparametrizations of the same curve: $\gamma_n = \gamma$. Now each function Ψ_n maps U onto the same curve γ, and (9) implies that the *rate of change of the arguments* of $\Psi_n(x)$ is the same non-zero constant c. We conclude that all Ψ_n are equal which implies that Ψ is a periodic function. According to Lemma 2 this can happen only if f is conjugated to z^d or to a Chebyshev polynomial or to a Lattès map. This proves our Theorem in the case that $c \neq 0$ and thus completes the proof.

2 Rational functions with real Julia sets

Proof of Theorem 2. Evidently $f(C) \subset C$. If there are no critical points on C, then the restriction $f : C \to C$ is a covering. The degree of this covering should be equal to $\deg f$ since every point of the Julia set has $\deg f$ preimages in C. Thus C is completely invariant and f^2 is a Blaschke product. From now on we assume that f has a critical point on C.

If $J(f) = C$ then both components of the complement of C are invariant under f^2, so f^2 is a Blaschke product in this case as well.

If $J(f) \neq C$, the set of normality is connected thus there is a fixed point z_0 to which the iterates on the set of normality converge. As $f(C) \subset C$, f commutes with reflection with respect to C. This implies that $z_0 \in C$. Evidently, the multiplier of z_0 satisfies $-1 \leq \lambda \leq 1$.

We may assume without loss of generality that $C = \bar{\mathbb{R}}$, and $z_0 = \infty$. Let $I = [a, b]$ be the convex hull of the Julia set. This means that $\bar{\mathbb{R}} \backslash I$ is the immediate basin of attraction of ∞ for the restriction $f|_{\bar{\mathbb{R}}}$. As the boundary of the immediate basin is invariant, we obtain

Lemma 6. *The set $\{a, b\}$ is f-invariant. If $f([a, b]) \subset [a, b]$ then $J(f) = [a, b]$.*

If $f([a,b]) \not\subset [a,b]$ then there exists an interval in $(\alpha,\beta) \subset [a,b]$ which is mapped by f outside $[a,b]$ (and α,β are mapped into a or b. Since the preimages of α,β are dense in the Julia set, it follows that in this case the Julia set if a Cantor set.

Lemma 7. *Each critical points of f in I is contained in the closure of a real interval which is component of the basin of x_0. In particular, each critical point in the Julia set is pre-periodic.*

Proof. Let us call a point $x_0 \in J(f)$ an *endpoint* of $J(f)$ if $J(f)$ accumulates to x_0 only from one side (left or right). It is clear that the endpoints of $J(f)$ are boundary points of the basin of x_0. On the other hand, if x is a critical point and $c = f(x)$ the corresponding critical value then one of the equation $f(x) = c + \epsilon$ or $f(z) = c - \epsilon$ has non-real solutions in a neighborhood of x for all sufficiently small ϵ. Thus the critical value $c \in J$ has to be an endpoint of $J(f)$. □

So there exists for each critical points $x \in (a,b)$ of f an integer $N \geq 1$ with $f^N(x) \notin (a,b)$.

This proves Theorem 2.

2.1 Polynomials with real Julia sets

For polynomials with real Julia sets, a complete parametric description is possible.

Let f be a polynomial of degree d whose Julia set J is real. We may assume that the convex hull of J is $[0,1]$. Then all d zeros of f are real (belong to $[0,1]$). Thus all critical points are also real and belong to $[0,1]$. Let $c_1,\ldots,c_{d-1}$ be the critical values enumerated left to right. Then the condition that the equation $f(z) = 1$ has all solutions real implies that all c_j are outside the interval $(0,1)$. Moreover, we obtain for odd d that 0 and 1 are either fixed or make a 2-cycle. For even d we have $f(0) = f(1) \in \{0,1\}$.

Now, critical values of such polynomials satisfy

$$(-1)^j c_j \quad \text{is of constant sign.} \tag{10}$$

This solves the classification problem completely. We can prescribe *arbitrarily* $d-1$ critical values $c_j \in \mathbb{R}\backslash(0,1)$ satisfying (10). Then there exists a real polynomial with these critical values (ordered sequence!). This polynomial is unique up to the change of the independent variable $z \mapsto az + b$ with positive a and real b. Using this change of the variable we achieve that the convex hull of the set $\{z : f(z) \in \{0, 1\}\}$ is $\{0, 1\}$.

Thus there is a bijective correspondence between sequences of critical values $(c_1, \ldots, c_{d-1})$ satisfying $c_j \in \mathbb{R}\backslash(0, 1)$ and (10) and polynomials with the property that the convex hull of the Julia set is $[0, 1]$. Chebyshev polynomials correspond to the case $c_j \in \{0, 1\}$. All other polynomials of our class have Cantor Julia sets.

2.2 Rational functions of the class (iii) in Theorem 2

We were unable to give any classification of these functions, so we only give several examples.

Example 1. The simplest non-polynomial example of case (iii) is a perturbation of a quadratic polynomial. Consider $f(z) = (z^2 - 4)/(1 + cz)$ with $c \in \mathbb{R}$. If $|c| < 1$, this map has an attractor at ∞ with multiplier c. Note that $f'(z) = \dfrac{cz^2 + 2z + 4c}{(1 + cz)^2}$ and this has two real zeros when $-1/2 < c < 1/2$. To compute $f^{-1}(\mathbb{R})$, we note that $f(z) = w$ is equivalent to

$$z = \frac{cw \pm \sqrt{c^2w^2 + 4w + 16}}{2}.$$

It follows that when $|c| > 1/2$, $c \in \mathbb{R}$, then $f^{-1}(\mathbb{R}) \subset \mathbb{R}$ and so f is a Blaschke product, while for $|c| < 1/2$, $c \in \mathbb{R}$ we find $f^{-1}(\mathbb{R}) \not\subset \mathbb{R}$ and so f is not a Blaschke product. Note that f is a Blaschke product with an attracting fixed point at ∞ if $c \in \mathbb{R}$ and $1/2 < |c| < 1$.

As remarked, for $|c| < 1/2$, f is not a Blaschke product. Let us determine its Julia set. There exists an interval $I = [p, q]$ containing 0 so that $f(p) = f(q) = q$, $f\colon I \to \mathbb{R}$ is continuous and has a minimum at some $c \in \operatorname{int}(I)$ with $f(c) < p$. Hence there exists two disjoint intervals I_0, I_1 in I which are mapped diffeomorphically onto I and so f has a full horseshoe Λ in $[p, q]$. Each other point is in the basin of the attractor at ∞. Since f has degree two, this horseshoe is also backward invariant, $f^{-1}(\Lambda) = \Lambda$ and so it follows that $J(f) = \Lambda \subset \mathbb{R}$.

Example 2. A function of the type (iii) can have a neutral rational fixed point. Indeed, take $f(z) = \dfrac{(z-2)(z+c)(z-c)}{(z-1)(z+1)}$ with $c \in (0, 1)$ close to 1.

Then ∞ is a parabolic fixed point which attracts real points $x \in (-\infty, -1)$ and repels points with $x \in \mathbb{R}$ and x large. (Indeed, $f(x) < x$ for $x \in \mathbb{R}$ and $|x|$ large because $\frac{(x+c)(x-c)}{(x-1)(x+1)} > 1$ for $|x| > 1$ and therefore $f(x) < x$ when $x \in (-\infty, -1)$. A similarly argument shows that $f(x) < x$ when $x \in (1, \infty)$.) The map f has a unique minimum $c \in (-1, 1)$ with $f(c) < -1$. There are three disjoint intervals I_1, I_2, I_3 with $I_1, I_2 \subset (-1, 1)$ and $I_3 \subset (1, \infty)$ such that f maps each of these diffeomorphically onto $(-1, \infty)$. So the Julia set contains a set $\Lambda \subset (-1, \infty)$ on which f acts as subshift of three symbols. Since f has degree 3, it follows that each preimage of this interval again lies inside this interval. Hence $J(f) = \Lambda \subset \mathbb{R}$. Each point outside Λ is in the basin of ∞. Clearly f is not a Blaschke product (there exist critical points on $\mathbb{R}$ so $f^{-1}(\mathbb{R})$ is not contained in $\mathbb{R}$).

Our last example shows that in general one cannot take $N = 1$ in Case (iii) of Theorem 2.

Example 3. We begin with a Blaschke product of degree 2,

$$g(z) = Kz\frac{z-a}{z-p}, \quad 0 < p < a < 1,$$

where the constants are chosen such that $K > 1$, $f(1) = 1$, and $f'(1) > 1$. This function has two branches defined on subintervals of $[0, 1]$ that map each subinterval on the whole $[0, 1]$, so the Julia set is a Cantor set whose convex hull is $[0, 1]$. Fix a closed interval $I \subset (p, a)$ on which $f(x) \le -1$, (i.e. I is in the basin of the attractor at infinity), and let c be the middle point of this interval. Let b be the preimage of the point c on the interval $[a, 1]$. Now we make a small perturbation of g, so that the resulting rational function of degree 3 is very close to g on $[0, 1]$ minus a small neighborhood of the point b. Our function is

$$f(z) = K(\epsilon)g(z)\frac{z-b+\epsilon}{z-b-\epsilon},$$

where ϵ is a very small positive number, and $K(\epsilon)$ is chosen so that $f(1) = 1$, so that $K(\epsilon) \to 1$ as $\epsilon \to 0$. It is clear that f has two critical values $c_1 < c_2$ on $[0, 1]$ at the critical points near b.

It is also easy to see that these critical values both tend to c as $\epsilon \to 0$. (Indeed, fix $\delta > 0$ small and let $V = [b-\delta, b+\delta]$ be a small neighborhood of b. Our function f converges to g and also f' converges to g' outside V

(as $\epsilon \to 0$). In particular $f(b+\delta)$ is close to $c = g(b)$, and f is *increasing* at this point $b + \delta$. But f also has a pole at $b + \epsilon < b + \delta$, and it is *decreasing* on the right hand side of this pole. It follows that f has a critical point (a minimum) on the interval $[b+\epsilon, b+\delta]$ with critical value at most $g(b+\delta)+\delta$ (when $\epsilon > 0$ is close to zero), which is close to $c = g(b)$. There is also another critical point on the other side of b, where the critical value is greater than $g(b-\delta) - \delta$. As the right critical value is evidently greater than the left one, both critical values tend to $c = g(b)$.)

So if ϵ is small enough, we have $f([c_1, c_2]) \cap I =$, thus the whole interval $[c_1, c_2]$ escapes from $[0, 1]$ under the second iterate of f, and we conclude that $J(f) \subset [0, 1]$, because each point of $[0, 1]\backslash[c_1, c_2]$ has three preimages in $[0, 1]\backslash[c_1, c_2]$.

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