ON THE ZERO SETS OF CERTAIN ENTIRE FUNCTIONS

ALEXANDRE EREMENKO AND L. A. RUBEL

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Abstract. We consider the class $B$ of entire functions of the form

$$f = \sum_{j=0}^{\infty} p_j \exp g_j,$$

where $p_j$ are polynomials and $g_j$ are entire functions. We prove that the zero-set of such an $f$, if infinite, cannot be contained in a ray. But for every region containing the positive ray there is an example of $f \in B$ with infinite zero-set which is contained in this region.

Let $B$ be Borel’s class of entire functions of one complex variable that are finite sums of entire functions with only finitely many zeros (possibly none). Clearly $f \in B$ if and only if

(1) $$f = \sum_{j=0}^{n} p_j \exp g_j,$$

where the $p_j$ are polynomials and the $g_j$ are entire functions. This class is called $B_1$ in [HRS].

Theorem 1. No function in $B$ can have as its zero set an infinite set of positive real numbers.

Theorem 2. Given any open set $\Omega$ in the complex plane that contains the positive real axis, there is a function $f$ in $B$ whose zero set is an infinite subset of $\Omega$.

Proof of Theorem 1. We will use H. Cartan’s theory of holomorphic curves [C, L]. An $n+1$-vector of entire functions $(f_0, \ldots, f_n)$ without zeros common to all $f_j$ defines a holomorphic curve $F$ which is a holomorphic map of the complex plane $C$ into the complex projective space $P^n$. The characteristic $T(r, F)$ is defined in the following way:

$$T(r, F) = \frac{1}{2\pi} \int_0^{2\pi} \max (\log |f_0|, \ldots, \log |f_n|) (re^{i\theta}) d\theta.$$

For any vector $a = (a_0, \ldots, a_n) \in C^{n+1} \setminus \{0\}$ define

$$N(r, a, F) = \frac{1}{2\pi} \int_0^{2\pi} \log |a_0 f_0 + \ldots + a_n f_n| (re^{i\theta}) d\theta.$$
Such a vector \( \mathbf{a} \) defines a hyperplane in \( \mathbb{P}^n \) by the equation \( a_0 x_0 + \ldots + a_n x_n = 0 \). If we denote by \( n(r, \mathbf{a}, F) \) the number of preimages of this hyperplane under \( F \) which are contained in the disk \( \{ z : |z| \leq r \} \), then by the Jensen formula

\[
N(r, \mathbf{a}, F) = \int_0^r \{ n(t, \mathbf{a}, F) - n(0, \mathbf{a}, f) \} \frac{dt}{t} + n(0, \mathbf{a}, F) \log r + \text{const}.
\]

If \( n = 1 \), the Cartan characteristic \( T(r, F) \) coincides (up to an additive constant) with the usual Nevanlinna characteristic of the meromorphic function \( f_1/f_0 \). We will use the Second Main Theorem of Cartan, which (in a simplified form) states the following: Let \( \mathbf{a}_1, \ldots, \mathbf{a}_q \) be an admissible system of vectors; that is, any \( n + 1 \) of them are linearly independent. If the components \( f_0, \ldots, f_n \) of a curve \( F \) are linearly independent, then

\[
\sum_{j=1}^q N(r, \mathbf{a}_j, F) \geq (q - n - 1 + o(1)) T(r, F), \quad r \in \mathbb{R}^+ \backslash E,
\]

where \( E \) is an exceptional set of finite length.

The following theorem due to E. Borel (see, for example [L, p. 186]) is a simple corollary of the Second Main Theorem of Cartan. Let \( f_j = p_j \exp g_j \), where \( p_j \neq 0 \) are polynomials and \( g_j \) are entire functions. If \( \{f_0, \ldots, f_n\} \) are linearly dependent, then there are two functions \( \exp g_j \) and \( \exp g_k \), which are proportional (with constant coefficients).

It follows from Borel’s theorem that every function of the class \( \mathcal{B} \) can be written in reduced form, namely the functions \( f_j = p_j \exp g_j \) in (1) are linearly independent. Furthermore in the proof of Theorem 1 we may assume without loss of generality that \( f \) is transcendental, the polynomials \( p_j \) have no zeros common to all \( p_j \) and that \( g_0 = 0 \).

With these assumptions we introduce the holomorphic curve \( F \) with coordinates \( f_j = p_j \exp g_j, \ 0 \leq j \leq n \), and show first that

\[
r = O(T(r, F)), \quad r \to \infty.
\]

Because \( f \) in (1) is assumed to be transcendental, at least one of \( g_j \) is not constant. Assume that \( g_n \neq \text{const} \). Then by the definition of characteristic and by our assumption that \( g_0 = 0 \) we have

\[
2\pi T(r, F) \geq \int_0^{2\pi} \max\{\log |f_0|, \log |f_n|\} \, d\theta \\
\geq \int_0^{2\pi} \max\{0, \text{Re } g_n\} \, d\theta + O(\log r) \geq cr + O(\log r),
\]

for some \( c > 0 \), which proves (3).

We need the following estimate

\[
T(r, f) \leq T(r, F) + O(\log r), \quad r \to \infty.
\]
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To prove this we use first the inequality \( \log |a+b| \leq \max\{\log |a|, \log |b|\} + \log 2 \) and then our assumption that \( g_0 = 0 \) (so \( \log |f_0| = \log |p_0| = O(\log r) \));

\[
2\pi T(r, f) = \int_0^{2\pi} \log |f| d\theta
\]

\[
\leq \int_0^{2\pi} \max\{\log |f_0|, \ldots, \log |f_n|\}^+ d\theta + O(1)
\]

\[
= \int_0^{2\pi} \max\{0, \log |f_1|, \ldots, \log |f_n|\} d\theta + O(\log r)
\]

\[
\leq \int_0^{2\pi} \max\{\log |f_0|, \ldots, \log |f_n|\} d\theta + O(\log r)
\]

\[
= 2\pi T(r, F) + O(\log r).
\]

Now we apply the Second Main Theorem of Cartan with \( q = n+2 \), and the following vectors: \( a_j \) for \( 1 \leq j \leq n+1 \) is the \( j \)-th row of the \((n+1) \times (n+1)\) unit matrix and \( a_{n+2} = (1, \ldots, 1) \) is the row of 1's. Then we have \( N(r, a_j, F) = O(\log r) \), \( r \to \infty \), and \( N(r, a_{n+2}, F) = N(r, 0, f) \), the usual Nevanlinna counting function of zeros of the entire function \( f \). From (2) it follows that

\[
N(r, 0, f) \geq (1 + o(1))T(r, F), \quad r \in \mathbb{R}^+ \setminus E.
\]

Combined with (4) this implies

\[
N(r, 0, f) \sim T(r, f), \quad r \to \infty, \quad r \in \mathbb{R}^+ \setminus E.
\]

In particular, this asymptotic equality combined with (3) implies that the genus of \( f \) is at least 1 (maybe infinite).

Finally we use the following result of A. Edrei and W. H. J. Fuchs [EF] and J. Miles [M]: If \( f \) is an entire function of genus at least 1, with positive zeros, then there is a set \( E_1 \) of zero logarithmic density and a constant \( \epsilon > 0 \) such that

\[
N(r, 0, f) \leq (1 - \epsilon)T(r, f), \quad r \in \mathbb{R} \setminus E_1.
\]

Since this estimate is incompatible with (6), Theorem 1 must hold.

**Proof of Theorem 2.** By taking a smaller region if necessary (but still including the positive real axis), we may assume that \( \Omega \) is connected and simply connected, and is bounded by a single smooth simple curve \( \gamma : [-1, 1] \to \mathbb{C} \) such that \( \gamma(t) \to \infty \) as \( t \to \pm 1 \) and \( \gamma \) intersects the real axis once (this intersection happens on the negative ray). The complement \( T \) of \( \Omega \) is an Arakelyan set, i.e. \( \Omega \) is connected and locally connected at \( \infty \) (see [GAI]). Using the Arakelyan approximation theorem [GAI] we find a non-constant entire function \( g \) with the property \( |g(z) - 1/2| < 1/4, \ z \in T \). Thus \( g^{-1}(Z) \subset \Omega \) and \( f(z) = \exp[2\pi i g(z)] - 1 \) gives the required example.

**REFERENCES**


Department of Mathematics, Purdue University, West Lafayette, Indiana 47907
E-mail address: eremenko@math.purdue.edu

Department of Mathematics, University of Illinois, 1409 West Green Street, Urbana, Illinois 61801