Schwarzian derivatives of rational functions

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1. Properties of the Schwarzian derivative,

\[{y, z} = \frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2, \quad \text{where} \quad ' = \frac{d}{dz}.\]

1.1 Consider the second order linear differential equation

\[w'' + Gw = 0. \tag{1}\]

If \(w\) and \(w_1\) are two linearly independent solutions of (1), then \(y = w_1/w\) satisfies

\[{y, z} = 2G. \tag{2}\]

To verify this, we recall that the Wronskian \(w_1'w - w_1w' = c\) is constant. So we have

\[y = \frac{w_1}{w}, \quad y' = \frac{c}{w^2}, \quad y'' = -2c \frac{w'}{w^3},\]

and

\[y''' = -2c \frac{w''w - 3w'^2}{w^4},\]

so

\[{y, z} = \frac{y'''}{y'} - \frac{3}{2} \left( \frac{y''}{y'} \right)^2 = -2 \frac{w''}{w} = 2G.\]

In the opposite direction, if \(y\) satisfies (2) then

\[w = \frac{1}{\sqrt{y}} \quad \text{and} \quad w_1 = \frac{y}{\sqrt{y'}}\]

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satisfy (1), which is verified by direct computation.

1.2 As a corollary we obtain

\[ \{y_1, z\} = \{y_2, z\} \text{ if and only if } y_1 = L \circ y_2, \]

where \(L\) is a fractional-linear transformation.

1.3 If \(y\) is a meromorphic function of \(z\) then \(\{y, z\}\) is meromorphic, moreover, it is holomorphic except at the critical points of \(y\), where it has poles of exactly second order. This can be verified directly, or using 1.1.

2. Regular singular points of the equation (1). A point \(z_0\) is called singular if \(G\) is not holomorphic at \(z_0\). A singular point is regular (see [5, 6]) if \(G\) has a pole of order at most 2 at this point. Making the change of variable \(w(z) = v(1/z), G(z) = H(1/z), \) and \(\zeta = 1/z,\) we obtain

\[ \zeta^4 v'' + 2\zeta^3 v' + H(\zeta) v = 0, \tag{3} \]

and the singular point at \(\infty\) is regular iff

\[ G(z) = O(z^{-2}), \quad z \to \infty. \tag{4} \]

If one solution of (2) is meromorphic in some domain \(D\) then all solutions are meromorphic in \(D\) (because all solutions are obtained from one of them by a fractional-linear transformation), and all singularities of (1) in \(D\) are regular in this case.

2.1 Suppose that 0 is a regular singular point of the equation (1). To use the general theory of regular singular points (see, for example, [5, 6]) we write the equation in the form

\[ z^2 w'' + P(z) w = 0, \quad \text{where} \quad P(z) = a_0 + a_1 z + a_2 z^2 + \ldots. \tag{5} \]

Put \(F(r) = r(r-1) + a_0,\) this is called the characteristic polynomial of (5), corresponding to the point 0. Let \(r_1\) and \(r_2\) be the two solutions of the indicial equation

\[ F(r) = r(r-1) + a_0 = (r-r_1)(r-r_2) = 0. \tag{6} \]

The following cases may occur:
a) If \( r_1 - r_2 \) is not an integer, equation (5) has two linearly independent convergent power series solutions of the form \( w_j(z) = z^{r_j}Q_j(z), \ j = 1, 2. \) Here \( Q_j \) are McLauren series (containing only non-negative integral powers of \( z \)).

b) If \( r_1 - r_2 \) is an integer, then \( r_1 \) and \( r_2 \) are real, because \( r_1 + r_2 = 1 \) by Vieta’s theorem. We label them so that \( r_1 \geq r_2 \). Then, if \( r_1 - r_2 \neq 0 \), there are two linearly independent solutions of the form

\[
\begin{align*}
    w_1(z) &= z^{r_1}Q_1(z) \\
    w_2(z) &= z^{r_2}Q_2(z) + Cw_1(z) \log z,
\end{align*}
\]

where \( C \) is a constant, \( Q_1 \) and \( Q_2 \) are McLauren series. If \( r_1 = r_2 \) there are two linearly independent solutions of the form

\[
\begin{align*}
    w_1(z) &= z^{r_1}Q_1(z) \\
    w_2(z) &= w_1(z) \log z + z^{r_1}Q_2(z),
\end{align*}
\]

so a logarithm is always present in the general solution in this case.

2.2 We are interested in the case, when

\[
    r_1 = r_2 + 2, \quad \text{and} \quad C = 0. \tag{7}
\]

(The conditions that \( r_1 - r_2 \) is a positive integer, and \( C = 0 \) are necessary and sufficient for the ratio of two linearly independent solutions to be meromorphic at 0. The additional condition \( r_1 - r_2 = 2 \) ensures that this meromorphic ratio has a simple critical point at 0.)

The first of these conditions (7), together with \( r_1 + r_2 = 1 \), imply that

\[
\begin{align*}
    r_2 &= -1/2, \quad r_1 &= 3/2 \quad \text{and thus by (6)} \tag{8} \\
    a_0 &= P(0) = -3/4. \tag{9}
\end{align*}
\]

To find the necessary and sufficient condition for \( C = 0 \) in (7) in terms of coefficients \( a_j \) of \( P \) in (5), we plug the power series \( w(z) = z^r(c_0 + c_1z + \ldots) \) with \( r = r_2 = -1/2 \) and \( c_0 = 1 \) into (5), where \( a_0 = -3/4 \), according to (8). We obtain

\[
\begin{align*}
    r(r - 1) + a_0 &= 0, \tag{10} \\
    [(r + 1)r + a_0]c_1 &= -a_1c_0, \tag{11} \\
    [(r + 2)(r + 1) + a_0]c_2 &= -a_2c_0 - a_1c_1, \tag{12} \\
    \ldots,
\end{align*}
\]
and in general

\[ F(r + n)c_n = \text{polynomial in } c_0, \ldots, c_{n-1}, \quad n = 0, 1, 2, \ldots, \tag{13} \]

where \( F \) is defined in (6). Equation (10) is satisfied because \( r = -1/2, a_0 = -3/4 \). Equation (11) (with \( r = -1/2, a_0 = -3/4 \) and \( c_0 = 1 \)) then implies \( c_1 = a_1 \), and equation (12), whose left hand side is 0, implies

\[ a_1^2 + a_2 = 0. \tag{14} \]

Thus if (5) has a power series solutions with properties (7), then (9) and (14) are satisfied. The converse is also true: if these two conditions are satisfied, than all coefficients \( c_j \) can be successively found from (13), because \( F(r + n) \neq 0 \) for \( n \geq 3 \). Thus we have

**Proposition.** In order that the ratio of two linearly independent solutions of (5) be meromorphic at 0, and have a simple critical point there, it is necessary and sufficient that conditions (9) and (14) be satisfied.

3. **Schwarzian derivatives of rational functions.** We say that a finite critical point \( z_0 \) of a rational function \( f \) is simple if \( f''(z_0) \neq 0 \). (If \( f(z_0) = \infty \), this has to be modified to \( (1/f)''(z_0) \neq 0 \).)

**Theorem 1.** Suppose that \( f \) is a rational function whose finite critical points \( z_1, \ldots z_n \) are simple. Then

\[ \frac{1}{2} \{ f, z \} = -\frac{3}{4} \sum_{k=1}^{n} \frac{1}{(z - z_k)^2} + \sum_{k=1}^{n} \frac{x_k}{z - z_k}, \tag{15} \]

where

\[ x_m^2 + \sum_{k \neq m} \frac{x_k}{z_m - z_k} = \frac{3}{4} \sum_{k \neq m} \frac{1}{(z_m - z_k)^2}, \quad m = 1, \ldots, n. \tag{16} \]

**Remark.** Substitution

\[ x_m = y_m - \frac{1}{2} \sum_{k \neq m} \frac{1}{z_m - z_k} \]

simplifies equations (16) to

\[ y_m^2 = \sum_{k \neq m} \frac{y_k - y_m}{z_k - z_m}. \]
This equivalent form of equations (16) was obtained in [3].

**Proof.** Let $f$ be a rational function with simple finite critical points $z_1, \ldots, z_n$. Put $G = (1/2)\{f, z\}$. Then $G$ is a rational function with only poles at $z_k$, all these poles are double by 1.4, and $G$ satisfies (4). In particular, $G$ has the form

$$G(z) = \sum_{k=1}^{n} \frac{b_k}{(z - z_k)^2} + \frac{x_k}{z - z_k}.$$ 

Now the ratio of two linearly independent solutions of (1) with this $G$ has to be a rational function. An inspection of the cases a) and b) in section 2.1 shows that this will be the case only if at each singular point $z_k$ we have $r_1 - r_2$ an integer and $C = 0$ (so that there are no logarithms). From the additional condition that each $z_m$ is a simple critical point of $f$ we deduce that $r_1 - r_2 = 2$. Thus we have (9) with $a_0 = b_m$ and (14) with $a_0 = -3/4$, $a_1 = x_m$ for each singular point $z_m$. This gives $b_m = -3/4$ and (16).

**Theorem 2.** Let $z_1, \ldots, z_n$ be distinct complex numbers, $(x_1, \ldots, x_n)$ a solution of (16) and $G(z)$ the rational function defined by the right hand side of (15). Then:

$$\sum_{k=1}^{n} x_k = 0,$$

and

$$\sum_{k=1}^{n} x_k z_k - \frac{3}{4}n = \frac{1 - q^2}{4},$$

where $q$ is a positive integer.

Furthermore, the general solution of equation (11) with this $G$ is a rational function of degree $(n + q + 1)/2$ having simple critical points at $z_1, \ldots, z_n$ and a critical point of multiplicity $q - 1$ at infinity.

**Proof.** If (16) is satisfied then the differential equation (2) defines a meromorphic function $y$ in $C$. All critical points of $y$ in $C$ are simple and occur exactly at $z_1, \ldots, z_n$. By definition of $G$, we have

$$G(z) \sim c/z, \ z \to \infty.$$ 

Suppose first that $c \neq 0$. Then, by the well-known asymptotic analysis of the equation (1) (see, for example, [5]) we conclude that $y$ is a meromorphic function of order $1/2$ and has one asymptotic value. In addition, it has
finitely many critical points. It is clear that such function cannot exist. The conclusion is that \( c = 0 \), which is equivalent to (4). Computing this \( c \) from (15) we obtain (17).

**Remark.** We proved that (16) implies (17). Can one prove this fact in a more direct way?

As infinity is now a regular singular point of (1), we conclude that our meromorphic function \( y \) cannot have an essential singularity, so it is rational. This implies for the equation (3), that the difference between the exponents \( r_1 \) and \( r_2 \) is a positive integer, and there are no logarithms in the formal solutions. Let \( \lim_{z \to \infty} z^2 G(z) = a \). The indicial equation of (3) at infinity is

\[
r^2 + r + a = 0,
\]

and its solutions are

\[
r_1 = \frac{-1 + \sqrt{1 - 4a}}{2} \quad \text{and} \quad r_2 = \frac{-1 - \sqrt{1 - 4a}}{2}.
\]

Now \( q = r_1 - r_2 \) is a positive integer and we conclude that

\[
a = \frac{(1 - q^2)}{4}.
\]

Computing \( a \) from (15) we obtain (18).

The critical point of \( y \) at infinity has order \( q - 1 \), so the total number of critical points on the Riemann sphere is \( n + q - 1 \), so \( y \) has degree \( (n + q + 1)/2 \).

4. Let us call two rational functions \( f_1 \) and \( f_2 \) equivalent if \( f_1 = L \circ f_2 \) for some fractional-linear \( L \) by 1.2. Two rational functions are equivalent if they have the same Schwarzian derivative. An equivalence class contains a real function if and only if the Schwarzian derivative of functions of this class is real. Indeed, if there is a real function in a class than its Schwarzian derivative is real. In the opposite direction, suppose that the Schwarzian derivative \( G/2 \) of a class is real. Then the differential equation

\[
\{y, z\} = G/2
\]

has at least one real solution \( y_0 \) (take any real initial conditions to solve the Cauchy problem). This means that there is a real function in the class, namely \( y_0 \).
5. Suppose that \( q = 1 \) in (18). Then \( a = 0 \), and the condition of absence of logarithms in the formal solution at infinity gives

\[
\sum_{k=1}^{n} x_k z_k^2 = \frac{3}{2} \sum_{k=1}^{n} z_k.
\]

It is clear that \( q \leq n + 1 \), as a rational function cannot have more than half of its critical points at infinity. In the extremal case, \( q = n + 1 \), \( y \) is a “polynomial” (up to a fractional-linear transformation) of degree \( n + 1 \), and such solution of (16) is unique. The number of solutions with any fixed \( q \) can be counted using the method of [2], for example, it is the Catalan number for \( q = 1 \). In general, for a fixed \( \geq 2 \) and \( n \), it is the number of chord diagrams (degenerate nets, using the terminology of [2]) with \( n + 1 \) vertices on the unit circle, each vertex except one is the endpoint of exactly one chord, and the exceptional vertex is the endpoint of \( q - 2 \) chords. The sum of all these numbers, for \( 1 \leq q \leq n + 1 \) gives the total number of solutions of (16). If all \( z_k \) are real, all these solutions are real by the result of [1].

References


