On the linear differential equations whose general solutions are elementary entire functions

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The following system of algebraic equations was derived in [3], in the study of the linear differential equations whose all solutions are polynomials:

$$x_k^2 = \sum_{j \neq k} \frac{x_j - x_k}{a_j - a_k}, \quad k = 1, \ldots, n \tag{1}$$

Here $\mathbf{a} = (a_1 \ldots, a_n)$ is a vector of parameters with all coordinates distinct. It follows from the results of [2] (see also [3] where simpler proofs are given) that this system has the following properties:

(i) If all a_k are real, then all solutions $(x_1, \ldots, x_n)$ are real.

(ii) All solutions vary analytically, as functions of parameters in the region $D = \{\mathbf{a} : a_1 < a_2 < \ldots < a_n\} \subset \mathbf{R}^n$.

(iii) All solutions satisfy $x_1 + \ldots + x_n = 0$, and

$$(n+1)^2 - 4 \sum_{k=1}^{n} x_k a_k = s^2,$$

where s is an integer such that $n + s$ is odd.

Boris Shapiro, while experimenting with the system (1) on a computer, made the following empirical observations:

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a) Solutions occur with multiplicities s, where s is defined in (iii).

b) The number of solutions of multiplicity s is

$$\frac{2s}{n+s+1}\left(\begin{array}{c} n \\ (n+s-1)/2 \end{array}\right),$$

c) If the equation $x_1+\ldots+x_n$ is added to the system, then all solutions become simple.

Observation (b) can be visualized in the form of a table, which is called Catalan's triangle (with zeros):

n	$s=$ 1	2	3	4	5	6	7	8	9
0	1								
1	0	1							
2	1	0	1						
3	0	2	0	1					
4	2	0	3	0	1				
5	0	5	0	4	0	1			
6	5	0	9	0	5	0	1		
7	0	14	0	14	0	6	0	1	
8	14	0	28	0	20	0	7	0	1

The rows of this table correspond to $n=0,1,2\ldots$, and the columns to $s=1,2\ldots$. The numbers show how many distinct solutions of multiplicity s the system has. The non-zero entries of the first and second columns are Catalan numbers. The table has the property that each entry is the sum of two entries of the previous row, one on the left and one on the right, so it is similar to Pascal's triangle. This can be easily seen from the expression in b). It follows that the sum of multiplicities of all solutions for a given n is 2^n, as predicted by the Besout theorem.

The purpose of this paper is to explain Shapiro's observations a), b) and c). It turns out that the question is related to linear differential equations whose general solution is an elementary entire function, a sum of polynomials multiplied by exponentials.

Consider a linear differential equation

$$Ay''-A'y'+Cy=0, \quad \text{where} \quad A(z)=(z-a_1)\ldots(z-a_n), \tag{2}$$

with distinct a_k, and C is a polynomial, $\deg C \leq \deg A$.

If all solutions y are entire functions, then there is a basis of solutions of the form $(R_1(z)e^{\lambda z},\, R_2(z)e^{-\lambda z})$, where R_i are polynomials. This is a theorem of Halphen [5], see also [6, 15.5].

Suppose that A is a given polynomial of degree n with distinct roots, and let us try to find all polynomials C of degree at most n, with given leading coefficient $-c$, and such that there exists a pair of linearly independent entire solutions without common zeros. Let us denote

$$P(z) := -\frac{A'}{A} = \sum_{k=1}^{n} \frac{-1}{z-a_k}, \quad \text{and} \quad Q(z) = \frac{C}{A} = -c + \sum_{k=1}^{n} \frac{x_k}{z-a_k}. \tag{3}$$

Equation (2) has regular singularities at the points a_k, and the condition that there are two linearly independent entire solutions without common zeros implies that in a neighborhood of each singularity there exists a holomorphic solution y of the form

$$y(z) = 1 + c_1(z-a_k) + c_2(z-a_k)^2 + O(z-a_k)^3. \tag{4}$$

Writing the equation in the canonical form in a neighborhood of a_k, as

$$(z-a_k)y'' + P_k(z)y' + Q_k(z)y = 0, \tag{5}$$

where

$$P_k(z) = -1 + p_k(z-a_k) + O(z-a_k)^2, \quad \text{where} \quad p_k = \sum_{j\neq k} \frac{1}{a_j - a_k},$$

and

$$Q_k(z) = x_k + q_k(z-a_k) + O(z-a_k)^2, \quad \text{where} \quad q_k = -c - \sum_{j\neq k} \frac{x_j}{a_j - a_k},$$

and substituting (4) into (5), we obtain

$$c_1 = x_k,$$

and

$$p_k c_1 + x_k c_1 + q_k = 0, \tag{6}$$

so

$$x_k^2 = -p_k x_k - q_k.$$

Recalling the expressions for p_k, q_k, we obtain the following *necessary* condition for the equation (2) to have two linearly independent entire solutions without common zeros:

$$x_k^2 = \sum_{j\neq k} \frac{x_j - x_k}{a_j - a_k} + c, \quad k = 1, \ldots, n. \tag{7}$$

This is a generalization of (1) that we need.

Example 1 If $n = 1$, we have $x_1^2 = c$, so $x_1 = \pm\sqrt{c}$. Suppose that c is positive, and denote the positive square root of c by λ. Choosing $a_1 = 0$ we obtain two differential equations

$$zy'' - y' - (\lambda^2 z \pm \lambda)y = 0.$$

Taking the plus sign in parentheses we easily find two linearly independent solutions

$$y_1(z) = (2\lambda z - 1)e^{\lambda z}, \quad y_2(z) = e^{-\lambda z}.$$

Minus sign in parentheses gives a pair $(y_1(-z), y_2(-z))$. If $c = 0$, we have only one equation,

$$zy'' - y' = 0,$$

with a pair of linearly independent solutions $(z^2, 1)$.

The ratio $(y_1 + y_2)/(\lambda^2 y_2)$ is a meromorphic function

$$\lambda^{-2}\left((2\lambda z - 1)e^{2\lambda z} + 1\right),$$

which tends to $2z^2$ as $\lambda \to 0$.

Example 2 If $n = 2$, we take $a_1 = -1$ and $a_2 = 1$. Then

$$x_1^2 = (x_2 - x_1)/2 + c, \quad x_2^2 = (x_2 - x_1)/2 + c,$$

so $x_1^2 = x_2^2$. This gives us four solutions and four differential equations:

If $x_1 = x_2 = \pm\sqrt{c} = \pm\lambda$, then the differential equations are

$$(z^2 - 1)y'' - 2zy' - (\lambda z \pm 1)^2 y = 0.$$

If $x_1 = -x_2$, we obtain $x_1 = -1/2 \pm \sqrt{1/4 + \lambda^2}$, and the differential equations are

$$(z^2 - 1)y'' - 2zy' - (\lambda^2(z^2 - 1) + z \pm \sqrt{1 + 4\lambda^2})y = 0.$$

Proposition 1 *Condition* (7) *is necessary and sufficient for a differential equation* (2) *to have two linearly independent entire solutions of the form* $y(z) = R(z)e^{\lambda z}$, *where* $\lambda = \pm\sqrt{c}$, *and* R *a polynomial.*

In particular, condition (7) *with* $c = 0$ *is necessary and sufficient for the equation* (2) *to have a basis of polynomial solutions.*

Proof. It remains to prove sufficiency. From (5) we conclude that all singular points in $\mathbf{C}$ are regular, with exponents 0 and 2. Condition (6) guarantees that there is a power series solution corresponding to the smaller exponent. This implies that there are two linearly independent holomorphic solutions in a neighborhood of each singular point. Thus all solutions are entire functions. By the theorem of Halphen mentioned in the beginning, there is a basis of solutions of the form $R(z)e^{\lambda z}$. Substituting this form to the equation we find that $\lambda = \pm\sqrt{c}$. This proves the proposition. □

Proposition 2 *If* $c = 0$, *then every solution of the system* (7) *has the following properties:*

$$\sum_{k=1}^{n} x_k = 0, \tag{8}$$

$$(n+1)^2 - 4\sum_{k=1}^{n} x_k a_k = s^2, \tag{9}$$

where s *is an integer such that* $n + s$ *is odd, and* $1 \leq s \leq n + 1$. *In addition,*

if all a_k *are real then all* x_k *are real.* (10)

The integer s *is the local degree at infinity of the rational function* y_1/y_2.

This was proved in [3].

Comments. System (7) with $c = 0$ has a trivial solution $x_1 = \ldots = x_n = 0$ which corresponds to the polynomial of degree $n + 1$ with critical points $a_1, \ldots, a_n$. In this case $s = n + 1$. The opposite case is that $s = 1$ (possible only when n is even) and we have rational functions with n prescribed simple critical points. This case is characterized by the condition

$$q^* = (n^2 + 2n)/4.$$

Now we recall some relevant facts from [2, 3]. Let $c = 0$, and let all a_j be real. Solutions of (7) are parametrized by their *nets*. A net is the preimage of the real line under the real rational function $f = y_1/y_2$, where y_1 and y_2 are real linearly independent real solutions of (2). There is a natural equivalence relation on the nets. The points a_j are simple critical points of those rational functions f. If n is even, there are solutions of degree $n/2 + 1$ whose only critical points are the a_j. In addition, there are solutions which correspond to rational functions having a critical point at infinity. If n is odd, there is always a critical point at infinity. The critical point at infinity can be multiple. For given real $a_1, \ldots, a_n$, there is a unique solution with given net. So distinct solutions can be counted by counting the nets.

Now suppose that $c > 0$. Now solutions of the differential equation contain real exponentials $\exp \pm\sqrt{c}z$. Their ratios are meromorphic functions which still have all their critical points at a_j, all these critical points being simple. Each of these functions has two real asymptotic values, corresponding to the curves tending to infinity "to the right" and "to the left". Let us consider the nets of these meromorphic functions. They are like nets of the rational functions with two exceptions: the number of edges and faces is now infinite, and an edge can go from infinity to infinity. We introduce the following classification of the edges.

Edges beginning and ending on the real line, are of the *first kind.*

Edges going from a vertex on the real line to infinity are of the *second kind.*

Edges going from infinity to infinity are of the *third kind.*

Evidently, there are finitely many edges of the first and second kind, (no more than the number of critical points), and infinitely many edges of the third kind. Our meromorphic functions have two distinct real asymptotic values, α_1 and α_2, say α_1 on the left, and α_2 on the right.

Each end at infinity of each edge of the second or third kind is an asymptotic curve. So these ends can be labeled by asymptotic values. Clearly two ends of one edge of the third kind cannot be labeled by the same asymptotic value. (f is monotone on each edge!) Speaking a bit informally, every edge of the third kind goes from left to right. Denote by $s-1$ the number of edges of the second kind in the upper half-plane. Every edge of the second kind goes from its vertex on the real axis *either to the left or to the right*, that is on this edge f tends either to α_1 or to α_2 as $z \to \infty$.

Let $b_1 < \ldots < b_{s-1}$ be the vertices on the real line from which the edges $e_1, \ldots, e_{s-1}$ of the second kind begin. If some e_k goes to the left, then evidently all edges from b_j with $j < k$ go to the left as well. So we introduce the rightmost edge e_m of the second kind that goes to the left, it begins at

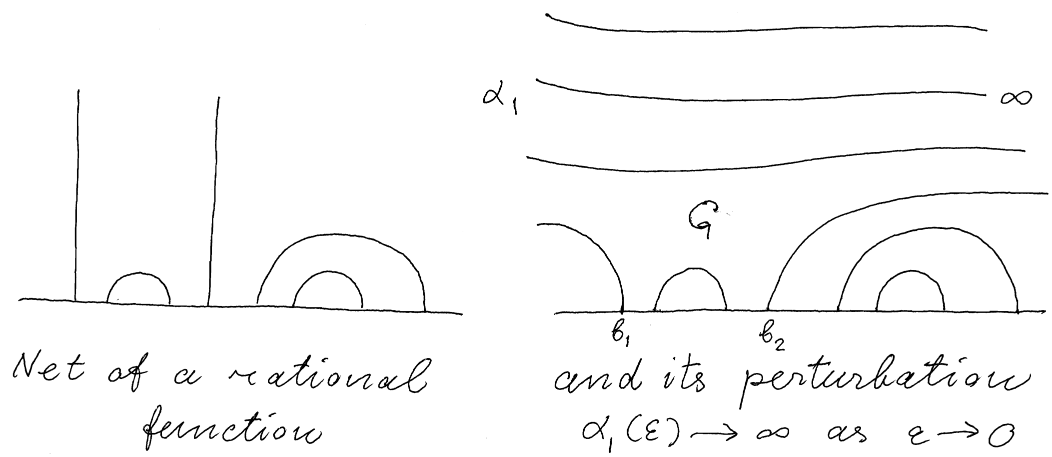

Figure 1:

b_m, where $0 \leq m \leq s-1$. All edges of the second kind on the right of e_m go to the right. This number m is the additional parameter which distinguishes the nets with given edges of the first and second kinds. The net is completely determined by its edges of the first and second kinds and the numbers m.

When c becomes small, our meromorphic function tends to the rational function. This rational function has edges of the first and second kind only (the edges of the third kind disappear at infinity. So exactly s transcendental nets collide to one rational net.

This explains the observed multiplicities and it remains to prove this.

Proposition 3 *For each rational function f_0 with n real simple critical points, and a point at infinity of multiplicity s, and for each sufficiently small positive c, there are exactly s classes of transcendental meromorphic functions close to f_0, of order one normal type, with n simple critical points close to the critical points of f_0.*

Proof. We use the construction from [2] (the simple part of that paper). To construct a rational function with given net, we used labeling of the net. Labeling is a function on the set of edges, with positive values, taking equal values on the edges symmetric with respect to the real line, and satisfying the following relations: for each face, the sum of the labellings of the boundary edges of this face is equal to 2π. For a rational function, the label of an edge is just the spherical length of its image.

Whenever any labeling of a net satisfying the conditions stated above is given, one can construct a function with this net and labeling, unique up to a fractional linear transformation of the independent variable, and up to a rotation of the sphere in the image. Similar construction can be performed in the transcendental case [9]. There is one substantial analytic difficulty in the transcendental case: the type problem. The construction of a meromorphic function with given net and labeling relies on the Uniformization Theorem. As for a transcendental function the Riemann surface in question is non-compact, there can be a priori two possible outcomes: a meromorphic function in the plane, or a meromorphic function in the unit disc. Fortunately, for our simple kind of nets this was solved: it is always the plane. Three different proofs are available, none of them simple: [8], [1] and [4]. This result of Nevanlinna not only says that the Riemann surface is of parabolic type but also that the uniformizing function satisfies a differential equation (2), that is it is a combination of polynomials with two exponentials.

Now we explain how to prescribe the labeling. Let p_0 be the labeling of the given rational function f_0, and suppose that our rational function is normalized somehow, using three points on the real line, but normalization and labeling imply that $f_0(\infty) = \infty$. Let $e_1, \ldots, e_{s-1}$ be the edges in the upper half-plane which go to infinity, e_0 the edge on the real line which goes to $-\infty$, and m a given integer in $[0, s-1]$. We modify the labeling of edges $e_0, \ldots, e_m$ only. The labeling of all other edges will remain the same, and normalization will remain the same. Namely we *shorten* the labels of $e_m, e_{m-2}, \ldots$, of every other edge beginning from e_m, by ϵ, and make the labels of $e_{m-1}, e_{m-3}, \ldots$ *longer* by ϵ. (So, for example, $p(e_0)$ becomes shorter if m is even, and longer if m is odd). Evidently this preserves the labeling condition for all faces except one, let's call is G: this is the one adjacent to e_m on the right. The sum over the boundary of this face will be $2\pi - \epsilon$. To compensate we add an edge of the third kind inside G, and label it ϵ. This new edge breaks G into two parts, let G_1 be that part whose boundary does not intersect the real line. We break G_1 further into infinitely many faces by adding infinitely many edges of the third kind, and label all these edges of the third kind by ϵ. This is the end of the construction.

By the result of Nevanlinna cited above, to any of these nets corresponds some meromorphic function $f_{m,\epsilon}$ which is of the form $f_{m,\epsilon}(z) = R_1(z)e^{\epsilon z} +$

$R_2(z)$, with rational functions R_1 and R_2.

Now it should be clear that for ϵ small enough we can achieve any prescribed position of the critical points near those of f_0. This proves the proposition.

$\square$

Now, applying the argument of [3], we obtain the following:

Theorem 4 *If a meromorphic function of order one, normal type has finitely many critical points, all of them real, then it is equivalent to a real function. All solutions of the system* (7) *with real* $\mathbf{a}$ *and (any!) positive* c *are real. There are exactly* 2^n *of them and they are all simple. For any fixed* $c > 0$*, the solutions vary analytically, as functions of* $\mathbf{a}$ *in the region* D*.*

This also explains observations a) and b) in the beginning. It remains to explain observation c). We will show that $c \neq 0$ implies $x_1 + x_2 + \ldots + x_n \neq 0$. Indeed, $x_1 + \ldots + x_n$ is the sum of the residues of the function

$$C/A = -y''/y + (y'/y)A'/A.$$

we substitute here a solution $y(z) = P(z)\exp(\lambda z)$, where P is a polynomial of the larger degree occurring in solutions, and $\lambda = \sqrt{c} \neq 0$. We obtain

$$C/A = -P''/P - 2\lambda P'/P + \lambda^2 + (P'/P + \lambda)(A'/A).$$

As all residues of P'/P and A'/A are 1, we conclude that the sum of the residues of C/A over finite poles is $\lambda n - 2d$, where $d = \deg P$. As $n \leq 2d - 2$ we conclude that the sum of the residues is not zero.

Remark. Of course, it is strange to use the hard analytic result of Nevanlinna in this situation, to prove a *local* result about an algebraic equation. It is desirable to find $f_{m,\epsilon}$ by some explicit perturbation of f_0 once we know the form of this perturbation. But I could only do this in Example 1 above.

2. System (1) can be generalized to m-tuples of linearly independent polynomials. We take $m = 3$.

Instead of (2) we obtain

$$Ay''' + By'' + Cy' + Dy = 0, \qquad (11)$$

where

$$A = -w(y_1, y_2, y_3), \quad B = -A', \quad D = w(y_1', y_2', y_3'),$$

and

$$C = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1'' & y_2'' & y_3'' \\ y_1''' & y_2''' & y_3''' \end{vmatrix}.$$

As earlier, we have

$$\deg B < \deg A, \quad \deg C < \deg A \quad \deg D < \deg A,$$

so denoting $P = B/A$, $Q = C/A$ and $R = D/A$ we obtain

$$Q(z) = \sum_{j=1}^{n} \frac{t_j}{z - a_j}, \quad R(z) = \sum_{j=1}^{n} \frac{x_j}{z - a_j}, \tag{12}$$

and P has the same expression as in (3).

Now we write the conditions on t_j and x_j which express the fact that all solutions of the differential equation

$$y''' + Py'' + Qy' + Ry = 0 \tag{13}$$

are polynomials. The derivation is similar to that done before. The resulting system is

$$\begin{aligned} x_k^2 &= \sum_{j \neq k} \frac{x_j - x_k}{a_j - a_k} - t_k \\ x_k t_k &= \sum_{j \neq k} \frac{t_j - t_k}{a_j - a_k}. \end{aligned}$$

For fourth order equations, the corresponding system is:

$$\begin{aligned} x_k^2 &= \sum_{j \neq k} \frac{x_j - x_k}{a_j - a_k} - 2t_k \\ x_k t_k &= \sum_{j \neq k} \frac{t_j - t_k}{a_j - a_k} - w_k \\ x_k w_k &= \sum_{j \neq k} \frac{w_j - w_k}{a_j - a_k}. \end{aligned}$$

Notice the peculiar feature of these systems: all equations, except the first group are linear with respect to x, t and w.

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