Electrostatic skeletons

Alexandre Eremenko[*], Erik Lundberg, and Koushik Ramachandran[†]

September 21, 2013

Dedicated to the memory of Andrei Gonchar and Herbert Stahl

Abstract

Let u be the equilibrium potential of a compact set K in $\mathbf{R}^n$. An electrostatic skeleton of K is a positive measure μ such that the closed support S of μ has connected complement and no interior, and the Newtonian (or logarithmic, when $n = 2$) potential of μ is equal to u near infinity. We prove the existence of an electrostatic skeleton for every convex polytope. Our proof raises several interesting questions on the uniqueness of a skeleton.

MSC: 31A05, 31A25. Key words: potential, equilibrium, subharmonic function, inverse problem, analytic continuation.

Let $K \subset \mathbf{R}^n$ be a compact set which is regular for the Dirichlet problem. The equilibrium potential u_K of K is a positive harmonic function in $\mathbf{R}^n \setminus K$ which satisfies

$$u_K(x) = 1 + O(1/|x|^{n-2}), \quad x \to \infty, \quad \text{for} \quad n > 2, \tag{1}$$

or

$$u_K(z) = \log|z| + O(1), \quad z \to \infty, \quad \text{for} \quad n = 2,$$

and whose boundary values on K are zero. Thus for $n = 2$ the equilibrium potential is the same as the Green function of $\mathbf{R}^2 \setminus K$ with pole at infinity. We use the unusual normalization in (1) to have a closer analogy between

[*]Supported by NSF grant.

[†]Partially supported by NSF grant DMS-1162070.

the cases $n = 2$ and $n > 2$: our equilibrium potentials, when extended by 0 on K, become subharmonic functions in $\mathbf{R}^n$.

A positive measure μ with closed support $S \subset K$ is called an *electrostatic skeleton* of K if S has empty interior, $\mathbf{R}^n \backslash S$ is connected, and

$$u_K(x) = 1 - \int \frac{1}{|x-y|^{n-2}} d\mu(y), \quad x \in \mathbf{R}^n \setminus K, \quad \text{for } n > 2,$$

or

$$u_K(z) = \int \log|z - \zeta| d\mu, \quad z \in \mathbf{R}^2 \backslash K, \quad \text{for } n = 2.$$

For example, an ellipsoid has an electrostatic skeleton supported on its $(n-1)$-dimensional focal ellipsoid [7].

For $n > 2$, if

$$P(x) = \sum_{i=k}^{d} \frac{a_k}{|x - x_k|^{n-2}}, \quad a_k > 0,$$

then the discrete measure with atoms of mass a_k at each point x_k is an electrostatic skeleton for the level sets $\{x : P(x) \geq L\}$, $L > 0$. The special case of this for $n = 2$ is that polynomial lemniscates have skeletons supported on finite sets.

An electrostatic skeleton of a special region bounded by two circles in $\mathbf{R}^2$ appears in [4]. It is clear that there are sets K which do not have an electrostatic skeleton. For example, a Jordan curve in $\mathbf{R}^2$ whose boundary is nowhere analytic. If u is the equilibrium potential of such a K, then any level set $\{z : u(z) < c\}$ also does not have an electrostatic skeleton.

Motivated by his study of the asymptotic behavior of zeros of Bergman (area orthogonal) polynomials, E. B. Saff proposed the problem about the existence of the skeletons in 2003 and mentioned it at several conferences (see also the reference to Saff in a recent paper by Lundberg and Totik [8]).

In particular, Saff asked whether every convex polygon has an electrostatic skeleton. Our Theorem 1 shows that this is so.

An analogous question with potentials of the volume or surface area measure instead of the equilibrium potential was considered by B. Gustafsson [3].

Theorem 1. *For every convex polytope $K \subset \mathbf{R}^n$, there exists an electrostatic skeleton with connected support.*

We will see in the proof that the intersection of the skeleton with int K is a semi-analytic set of codimension 1 [6, 1], in particular S has zero volume.

Lemma 1. *Let $V \subset \mathbf{R}^n$ be an open ball. Let u_1 and u_2 be two harmonic functions in V, and suppose that the set $E = \{x \in V : u_1(x) = u_2(x)\}$ is a non-singular analytic hypersurface which divides V into two regions D_1 and D_2. Let*

$$v(x) = \begin{cases} u_1(x), & x \in D_1, \\ u_2(x), & x \in D_2. \end{cases}$$

Then v is subharmonic if and only if $v(x) = \max\{u_1(x), u_2(x)\}$.

This is well-known and easy to prove.

Proof of Theorem 1. Let $L_j,\ j = 1, \ldots, N$ be the open $(n-1)$-faces of K, and ℓ_j the reflection in L_j. Then the equilibrium potential $u = u_K$ extends by reflection:

$$u_j = -u \circ \ell_j, \quad j = 1, \ldots, N.$$

As K is convex, these functions u_j are negative harmonic functions in the interior of K. The boundary values of u_j on ∂K are zero on the closure $\overline{L_j}$ and strictly negative on $K \backslash \overline{L_j}$. It follows that the u_j are pairwise distinct negative harmonic functions in int K.

So there is no open set $V \subset K$ where $u_j(x) = u_k(x)$ for $x \in V$ and $j \neq k$.

Consider the set W consisting of all subharmonic functions w in $\mathbf{R}^n$ with the following properties:

$$w(x) = u(x), \quad x \in \mathbf{R}^n \backslash K,$$

and

$$w(x) \in \{u_1(x), \ldots, u_N(x)\}, \quad x \in K.$$

To show that $W \neq \emptyset$, we present a function $w_0 \in W$:

$$w_0(x) = \begin{cases} u(x), & x \in \mathbf{R}^n \backslash K, \\ \max\{u_1(x), \ldots, u_N(x)\}, & x \in K. \end{cases}$$

Subharmonicity has to be verified only on ∂K. Functions u and u_k are analytic continuations of each other and they coincide with some harmonic function h in a neighborhood of L_k, while for every $j \neq k$, $u_j(x) < h(x)$ in a neighborhood of L_k. Thus $w_0(x) = h(x)$ in a neighborhood of L_j, so w_0 is harmonic on this neighborhood. The rest of the boundary $\partial K \backslash \cup L_j$

has codimension 2, so it is removable for bounded subharmonic functions [2]. This proves that $w_0 \in W$, so $W \neq \emptyset$.

For a function $w \in W$ we denote by $S[w]$ the support of the Riesz measure. It is clear that this support belongs to the set

$$E = \{x : \exists (i,j), i \neq j, u_i(x) = u_j(x)\}.$$

This is a semi-analytic subset of int K which is nowhere dense because all u_j are pairwise distinct harmonic functions. To every component D of int $K \backslash S[w]$ corresponds a number $k(D,w)$ such that $w(x) = u_k(x),\ x \in D$.

The set W is equipped with pointwise partial order: $w_1 \leq w_2$ means $w_1(x) \leq w_2(x),\ x \in \mathbf{R}^n$. If $A \subset W$ is a totally ordered subset, then there exists a lower bound w_A of A which is defined as $w_A(x) = \inf\{w(x) : w \in A\}$. It is clear that w_A defined in such a way belongs to W: the limit of a decreasing family of subharmonic functions is always subharmonic (if it is finite at least at one point), and the set W is evidently closed under pointwise convergence.

Thus Zorn's Lemma[1] implies that W contains minimal elements.

Let w_m be a minimal element of W. We will prove that $S[w_m]$ does not divide $\mathbf{R}^n$. Proving this by contradiction, we suppose that there is a bounded component D of $\mathbf{R}^n \backslash S[w_m]$. There is a number $k = k(D, w_m)$ such that $w_m(x) = u_k(x),\ x \in D$. Let

$$I = \{j : u_j(x) < u_k(x),\ x \in D\}.$$

Notice that $u_j(x_0) \leq u_k(x_0)$ for some $x_0 \in D$ implies that $j \in I$. This follows from the Maximum principle.

A priori, this set I could be empty, but we will show that it is not empty shortly. We define the function

$$v(x) = \begin{cases} w_m(x), & x \in \mathbf{R}^n \backslash D, \\ \max_{j \in I} u_j(x), & x \in D. \end{cases}$$

Let us show that this function is well-defined (that is $I \neq \emptyset$) and subharmonic.

Again, the subharmonicity has to be verified only on ∂D. We use the fact that every semi-analytic set can be decomposed into strata [6, 1]. Each stratum is an embedded analytic submanifold. Let $E = \cup_{i=0}^{n-1} E_i$ be this

[1] It is very plausible that the set W is finite for every convex polytope K, and this is the case when $n = 2$, but we could not find a simple proof of this for $n > 2$.

decomposition, where i is the dimension of E_i. Consider a point $x_0 \in \partial D \cap E_{n-1}$. There is an open neighborhood V of x_0 which is divided by E_{n-1} into two parts, $D_1 = V \cap D$ and $D_2 = V \setminus \overline{D}$. As w_m is harmonic in D_2, we have $w_m(x) = u_\ell(x),\ x \in D_2$ for some $\ell \neq k$. Then it follows by Lemma 1 that $w_m(x) = \max\{u_k(x), u_\ell(x)\},\ x \in V$, and $u_\ell(x) < u_k(x),\ x \in D_1$ and thus the same holds for $x \in D$. So $\ell \in I$. In particular, this shows that $I \neq \emptyset$. Thus $v(x) \geq u_\ell(x),\ x \in V$ and as $v(x) = u_l(x),\ x \in V \cap E_{n-1}$, we conclude that v is subharmonic in V. So $v \in W$. Evidently $v \leq w_m$ and $v \neq w_m$, which contradicts minimality of v_m.

This contradiction completes the existence proof of the skeleton $S = S[w_m]$.

It remains to prove that S is connected. Suppose that $S[w_m]$ is disconnected. Then there is a bounded region D such that $\partial D \cap S[w_m] = \emptyset$ and D contains a non-empty closed part S_1 of $S[w_m]$. Moreover, there is no relatively compact component of $D \setminus S_1$ in D which means that one of our functions u_k coincides with w_m on a dense subset of D. Then evidently $w(x) = u_k(x),\ x \in D$ and this is a contradiction.

Remarks and conjectures.

1. It is easy to see that a non-convex (Jordan) polygon in $\mathbf{R}^2$ cannot have a skeleton.

2. Convexity of K was used only to assure that an immediate analytic continuation of u through each side is a negative harmonic function in int K. We state as a separate theorem

Theorem 2. *Let K be a Jordan region, that is ∂K is an embedded sphere, and $\{I_j\}$ are disjoint open subsets of the boundary ∂K such that $\partial K \setminus \cup_j I_j$ has zero capacity. Suppose that the equilibrium potential u of K has an immediate analytic continuation through each I_j and that the resulting functions u_j are harmonic in the interior of K and strictly negative on $\partial K \setminus I_j$. Then there exists an electrostatic skeleton of K.*

A further generalization can be obtained: each u_j can be allowed to have an isolated singularity $a \in K$ such that $u_j(x) \sim -c|x-a|^{-n+2},\ x \to a$, $c > 0$ (or in the plane, $u_j(z) \sim c \log|z-a|,\ z \to a$). This generalization does not require any changes in the proof.

Corollary. *Let K be a circular polygon in $\mathbf{R}^2$, such that the reflection of $\overline{\mathbf{C}} \setminus K$ in each side I_j contains $K \setminus I_j$. Then K has an electrostatic skeleton.*

Now we give an example of a circular triangle in $\mathbf{R}^2$ which does not have an electrostatic skeleton. Let U be the unit disc, and $D \subset \Delta = \overline{\mathbf{C}}\backslash\overline{U}$ the hyperbolic triangle with vertices at $1, \exp(\pm 2\pi i/3)$. Then repeated reflections of D in the sides tile Δ. A more familiar picture of this tiling is obtained by changing the variable to $1/z$. Therefore the equilibrium potential has an analytic continuation to $\Delta\backslash E$, where E is a discrete set of logarithmic singularities. As these singularities are dense on the unit circle, $K = \mathbf{C}\backslash D$ cannot have an electrostatic skeleton.

3. We conjecture that the skeleton S is uniquely defined by the original polytope K, but we do not know how to prove this, except for the case that K is a simplex (see Theorem 3 below).

In general, the electrostatic skeleton of a Jordan region K does not have to be unique. An example of non-uniqueness was constructed by Zidarov [10], it is also reproduced in [3]. For simplicity we only discuss this example for $n = 2$. Zidarov constructed two distinct probability measures whose supports are trees consisting of finitely many straight line segments, and their potentials are equal in a neighborhood of ∞ to the potential of the area measure of a non-convex polygon. The level sets of these potentials $\{z : u(z) \leq c\}$ with sufficiently large c are Jordan regions with non-unique skeletons.

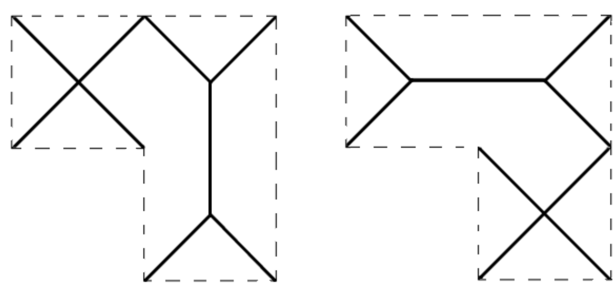

Figure 1: Two different trees supporting positive measures whose potentials coincide near infinity.

4. We conjecture that the support of the Riesz measure of w_0 in the proof of Theorem 1 does not divide the space. That is w_0 is a minimal element of W. If this is so, conjecture 3 would easily follow (Proposition 1 below).

5. When the equilibrium potential in the definition of skeleton is replaced by the volume measure on K, the resulting notion is called a "mother body". Gustafsson [3] proved existence and uniqueness of the mother body for every convex polytope in $\mathbf{R}^n$.

Next we show, using the notation from the proof of Theorem 1, that Conjecture 4 implies Conjecture 3.

Proposition 1. *In the proof of Theorem 1, if $S[w_0]$ does not divide $\mathbf{R}^n$, then the electrostatic skeleton is unique, and its closed support is $S[w_0]$.*

Proof. The existence of the electrostatic skeleton has been proved in Theorem 1. Let S be its support, and v the potential of the skeleton. Clearly $S \subset K$. By assumption, $S[w_0]$ does not divide $\mathbf{R}^n$. Then the complement int $K \backslash S[w_0]$ consists of N components, D_j, $1 \leq j \leq N$, where N is the number of $(n-1)$-faces of ∂K, such that D_j contains L_j.

As S is closed and has empty interior, the set $G = \text{int } K \backslash S$ is open and non-empty. Let p be a point in G. As S does not divide the plane, there is a curve γ in $\mathbf{C} \backslash S$ starting from p and ending outside K, and v has an analytic continuation on this curve. Consider the point q where γ leaves K for the first time, and let $q \in L_k$ (it is clear that q belong to a face of dimension $n-2$ of K). As u_k is the immediate analytic continuation of u to the interior of K, we conclude that $v(x) = u_k(x)$ in a neighborhood of p. Therefore

$$v(x) \in \{u_1(x), \ldots, u_N(x)\} \quad \text{for every} \quad x \in K, \tag{2}$$

so $v \in W$, the class defined in in the Proof of Theorem 1. In particular, v is continuous. Recall that $u_k(x) > u_j(x)$ for $x \in D_k$ and all $j \neq k$. As v is continuous, it follows from (2) that $v(x) = u_k(x)$ for $x \in D_k$. So $v = w_0$ by continuity, and $S = S[w_0]$.

Theorem 3. *If K is a simplex, then $S[w_0]$ does not divide $\mathbf{R}^n$, and there exists a unique electrostatic skeleton.*

Proof. Suppose that $\mathbf{R}^n \backslash S[w_0]$ has a bounded component D, then $D \subset K$, and let a be a point in D. Suppose without loss of generality that $k(D) = 1$, that is $u_1(x) > u_j(x),\ z \in D,\ j \in \{2, 3, .., n+1\}$. Let R be the straight line segment connecting a point $b \in L_1$ with the vertex opposite to L_1 which passes through a. Let $R_2, R_3, .., R_{n+1}$ be reflections of R with respect to the other faces $L_2, L_3, .., L_{n+1}$. These segments all lie outside K. Introduce the direction on $R, R_2, R_3, .., R_{n+1}$ from their common vertex. See Fig. 1. We

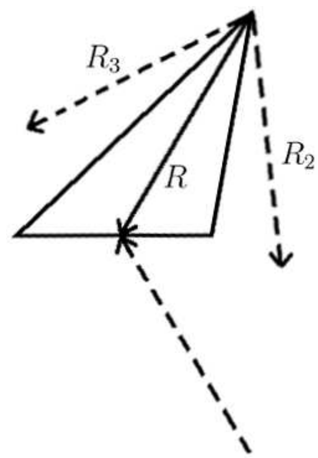

Figure 2: Construction for the proof of Theorem 2.

claim that our original equilibrium potential u is strictly increasing on these segments.

This follows from the fact that level hypersurfaces of the equilibrium potential of a convex set are convex (for $n = 2$, see [9], and for $n > 2$, [5]), and a straight line can cross a convex hypersurface at most twice; if the straight line segment begins inside a convex hypersurface and ends outside, then it crosses the boundary only once. Therefore, our segments $R_2, R_3, .., R_{n+1}$ cross each level surface at most once, so the function u is increasing on $R_2, R_3, .., R_{n+1}$. Therefore, $u_j = -u \circ \ell_j$ with $j \in \{2, 3, .., n+1\}$ are decreasing on R, while u_1 is increasing on R. So as we have $u_1(x) > \max\{u_2(x), u_3(x), .., u_{n+1}(x)\}$ for $x = a$ and $x = b$ we conclude that this inequality holds on the whole segment $[a, b] \subset R$ giving a contradiction which proves that S_1 does not divide the plane.

Now take $n = 2$. We show that there exists a rectangle K with sides L_j, and four negative harmonic functions u_j in K, such that $u_j(z) = 0,\ z \in L_j$, and the boundary values of u_j are negative on the other three sides, and there exist two distinct functions $w \in W$, whose Riesz measures are supported on trees. We recall that W consists of subharmonic functions w such that $w(z) \in \{u_1(z), \ldots, u_4(z)\}$ and zero boundary values on ∂K. Thus the set W defined in the proof of Theorem 1 may have several minimal elements in such setting.

Example 1. Let $K = [-1, 1] \times [-a, a]$, where $a > 0$ is a parameter to be specified later. Let the sides of K be called L_j, $1 \leq j \leq 4$, enumerated counter-clockwise, with $L_1 = [-1 - ai, 1 - ai]$.

Let $\phi(x) = 4x^3 - x - 3, \quad x \in [-1, 1]$. This function has the following properties: $\phi(1) = 0$, $\phi(x) < 0$ on $[-1, 1)$, it has two extrema a local maximum at $x = -1/\sqrt{12}$ and a local minimum at $x = 1/\sqrt{12}$. The equation $\phi(x) = \phi(-x)$ has three solutions $0, \pm 1/2$.

Fig. 2 shows the graphs of $\phi(x)$ and $\phi(-x)$ on $[-1, 1]$. Let u_2 be the

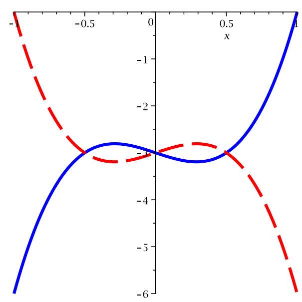

Figure 3: Graphs of $\phi(x)$ and $\phi(-x)$.

solution of the Dirichlet problem for K with boundary values $\phi(x)$ on both horizontal sides, 0 on L_2, and $-6 = \phi(-1)$ on L_4. Let $u_4(z) = u_2(-z)$. It is easy to see that when the parameter a is sufficiently small, the zero set of $u_4 - u_2$ consists of three simple curves which are close to the intervals

$$[-1/2 - ia, -1/2 + ia], \quad [-ia, ia], \quad [1/2 - ia, 1/2 + ia].$$

These three components of the zero set divide the rectangle into four regions, which we call G_1, G_2, G_3, G_4, enumerated left to right.

Now we define $u_1(z) = -M(y + a)$ and $u_3(z) = -M(a - y) = u_1(-z)$, where $z = x + iy$, and $M > 0$. The condition on M is that

$$u_1(x) = u_3(x) = -Ma < \max\{u_2(x), u_4(x)\} \quad \text{for} \quad -1 \leq x \leq 1.$$

With this arrangement, the set S_1 looks as in Fig. 3, where the numbers $k(D)$ indicate which function is maximal in each region. Applying the procedure of the proof of Theorem 1 to the region with $k(D) = 2$, we obtain a tree as in Fig. 4, and applying this procedure to the region when $k(D) = 4$, we obtain a tree as in Fig. 5. It is evident that these trees are distinct subsets of K.

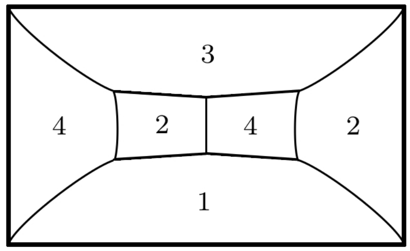

Figure 4: The set S_1.

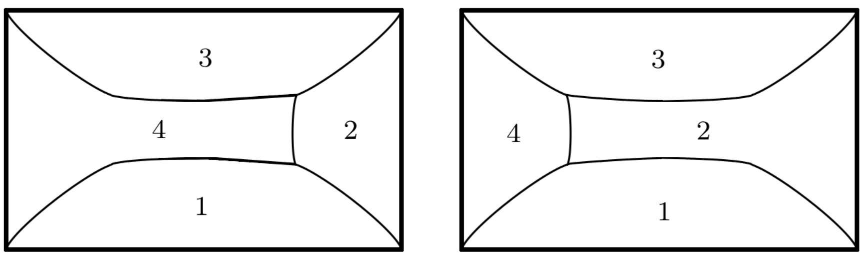

Figure 5: Result of removal of the left and right bounded components (respectively).

We thank Edward Saff and Vilmos Totik for an interesting discussion of this paper.

References

[1] E. Bierstone and P. Milman, Semianalytic and subanalytic sets, Inst. Hautes Études Sci. Publ. Math. No. 67 (1988), 5–42.

[2] L. Carleson, Selected problems on exceptional sets, Van Nostrand, Princeton NJ, 1967.

[3] B. Gustafsson, On mother bodies of convex polyhedra, SIAM J. Math. Anal., 29 (1998) 1106–1117.

[4] A. Levin, E. Saff, and N. Stylianopoulos, Zero distribution of Bergman orthogonal polynomials for certain planar domains, Constr. Approx. 19 (2003), 411–435.

[5] J. Lewis, Capacitary functions on rings, Archive for Rational Mechanics and Analysis, 66 (1977), 201–224.

[6] S. Łoyasjewicz, Une propriété topologique des sous-ensembles analytiques réels, Les Équations aux Dérivées Partielles (Paris, 1962) p. 87–89. Éditions du Centre National de la Recherche Scientifique, Paris, 1963.

[7] E. Lundberg and D. Khavinson, A tale of ellipsoids in potential theory, arXiv:1309.2042.

[8] E. Lundberg and V. Totik, Lemniscate growth, Anal. Math. Phys. 3 (2013), 1, 45–62.

[9] Ch. Pommerenke, Univalent functions, Vandenhoeck & Ruprecht, Göttingen, 1975.

[10] D. Zidarov, Inverse gravimetric problem in geoprospecting and geodesy, Elsevier, Amsterdam, 1990.

Department of Mathematics
Purdue University
West Lafayette IN 47907-2067
eremenko@math.purdue.edu
elundber@math.purdue.edu
kramacha@math.purdue.edu