Midterm exam 1 solutions

1a. 
\[
\begin{pmatrix}
1 & -1 & 1 & 3 \\
1 & -3 & 1 & 4 \\
5 & 0 & 2 & 5 \\
\end{pmatrix}
\]

1b. 
\[
\begin{pmatrix}
1 & 0 & 2/5 & 1 \\
0 & 1 & -1/5 & -1 \\
0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

1c. 
\[
\begin{pmatrix}
1 - (2/5)t \\
-1 + t/5 \\
t
\end{pmatrix}
\]
where \( t \) is any real number.

How to check: substitute this to the system.

2a. 
\[
\begin{pmatrix}
-1/2 & 1/2 & 1/2 \\
3/2 & -5/2 & 1/2 \\
-1/2 & 3/2 & -1/2 \\
\end{pmatrix}
\]
Checking: multiply this by \( A \).

2b. Compute the determinant. The matrix is singular when the determinant is 0:

\[
x^3 - 9x = 0
\]

Answer: \( x = 0, 3 \) or \(-3\).

2c. The adjoint matrix is
\[
\begin{pmatrix}
-1 & 1 & 1 \\
3 & -5 & 1 \\
-1 & 3 & -1 \\
\end{pmatrix}
\]
Checking: in a) we found the inverse matrix. Adjoint matrix is the inverse matrix times the determinant of \( A \).
3a. \(-24, -55\). Checking: compute it two different ways.

4a. This substitution is positive. In fact it is one cycle \((1, 3, 4, 5, 2)\) of length 5.

4b. There are \(6! = 720\) substitutions of degree 6, half of them even, another odd. So there are 360 odd substitutions.

5. a) False. Almost any pair of \(2 \times 2\) matrices will be a counterexample.
   b) True. \(\det A^3 = (\det A)^3\).
   c) False. Take \(A = 0\) and \(b \neq 0\).
   d) True. For example, \(x = 0\).
   e) False. \(I\) and \(2I\) have the same rref, namely \(I\) but \(\det (2I) = 2^n\).