Entire functions with radially distributed a-points

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Zeros of an entire function can be arbitrarily prescribed, but one cannot simultaneously prescribe zeros and 1-points.

We will study the simple case when zeros and 1-points belong to a finite set of rays from the origin. The principal result about such functions was proved by A. Edrei in 1955.

Theorem A. *Suppose that all zeros and* 1*-points of an entire function belong to a finite set of rays from the origin, and let* ω *be the smallest angle between these rays. Then* f *is at most of order* π/ω*, normal type.*

It is remarkable that a strong growth restriction comes only from the assumption on arguments of a-points. The deepest part of this theorem is that the order is finite. If one makes this an á priori assumption, then the result that the growth is at most of order π/ω, normal type, is much simpler and was already known to Bieberbach in 1919.

When we have only two rays, and they are positive and negative rays of the real line, Fuchs's theorem says that the function must be of exponential type. This leads to the famous parametric description of real entire functions whose all ± 1-points are real in the work of Marchenko and Ostrovskii (1975). Such functions occur in spectral theory as Hill discriminants.

Let

$$y'' + q(x)y = \lambda y$$

be a linear differential equation with real periodic coefficient, $q(x) = q(x+T)$. Then the shift operator $S : y(x) \mapsto y(x+T)$ acts on the space of real solutions and has determinant 1. The trace of the shift operator is a real entire function

$\lambda \mapsto 2L(\lambda)$. So the characteristic equation of S has the form

$$\rho^2 - 2L(\lambda)\rho + 1 = 0.$$

So $L(\lambda) = \pm 1$ if and only if $\rho = \pm 1$, that is λ is an eigenvalue of (4) with periodic or anti-periodic boundary conditions. It follows that all such λ are real because both eigenvalue problems are self-adjoint.

Returning to the theory of entire functions, we mention the following three theorems.

Theorem B. (Milloux, 1927) *If all zeros of an entire function f belong to a line L_0 and all 1-points to a different line L_1 which crosses L_0, then $f(z) = e^{az+b}$ or $f(z) = 1 + e^{az+b}$, or a polynomial of degree at most 2.*

Theorem C. (T. Kobayashi, 1979, I. N. Baker, 1980) *If all zeros of a transcendental entire function f belong to a line L_0 and 1-points to a parallel line $L_1 \neq L_0$, then $f = P(e^{az})$ with a polynomial P, and all such polynomials are explicitly described.*

Theorem D. (Biernacki, 1929) *If arguments of zeros of a transcendental entire function accumulate only to α and arguments of 1 points accumulate only to β, then $\alpha = \beta$.*

We complement these results with the following

Theorem 1. (Bergweiler, Eremenko, Hinkkanen, 2016) *If all zeros of a transcendental entire function f lie on the positive ray L_0 and all 1-points on two rays L_j, $j \in \{\pm 1\}$ different from L_0, then he rays L_j make equal, acute angles with L_0.*

A more interesting question is whether there exist non-trivial examples of entire functions whose zeros and 1-points are radially distributed. (By trivial examples we mean functions of the form $f(z^n)$, where all zeros and 1-points of f lie on a line, for example $f(z) = \sin z$. Such functions f have been completely described by Marchenko and Ostrovskii.)

It turns out that there are non-trivial examples, and they are also closely connected to spectral theory!

Theorem 2. (Bergweiler, Eremenko, Hinkkanen, 2016) *Let*

$$L_j = \{re^{ij\alpha} : r > 0\}, \quad j \in \{\pm 1\}.$$

If $\alpha \in (0, \pi/3]$ or $\alpha = 2\pi/5$, then there exists a transcendental entire function whose all zeros lie on the positive ray and all 1-points on $L_1 \cup L_{-1}$.

It is not known whether such functions exist for $\alpha \in (\pi/3, \pi/2)$ except for $\alpha = 2\pi/5$. I will only describe the proof for the case $\alpha = 2\pi/5$; the case $\alpha \in (\pi/3, \pi/2)$ uses the same idea, but it is more complicated, and I will only briefly discuss it in the end.

Proof of Theorem 2 for $\alpha = 2\pi/5$. Consider the differential equations

$$-y'' + \left(z^3 + \lambda\right) y = 0, \tag{1}$$

where λ is a complex parameter.

Set

$$\omega = e^{2\pi i/5}.$$

Our equation has the following symmetry property: if $y_0(z, \lambda)$ is a solution, then

$$y_k(z, \lambda) = y_0(\omega^{-k} z, \omega^{2k} \lambda) \tag{2}$$

is also a solution.

All solutions are entire functions of two variables, z and λ. To describe their asymptotic behavior, we consider the sectors

$$S_0 = \{re^{it} : r > 0,\ |t| < \pi/5\}, \quad S_k = \omega^k S_0.$$

These are called the Stokes sectors. In each Stokes sector, there is a one-dimensional space of solutions which tend to zero exponentially (uniformly with respect to $\arg z$ in any smaller sector), while all other solutions grow exponentially. A solution which tends to zero in S_k is called subdominant in S_k. It must grow in the two adjacent sectors S_{k-1} and S_{k+1}.

Asymptotics of solutions are obtained by the well known Green–Liouville method (known as the WKB method to physicists). In particular, there is a unique solution which satisfies

$$y_0(z, \lambda) = (1 + o(1)) z^{-3/4} \exp\left(-\frac{2}{5} z^{5/2}\right) \quad z \in S_{-1} \cup S_0 \cup S_1, \quad z \to \infty, \tag{3}$$

which we call the Sibuya solution. Notice that this asymptotics does not depend on λ. Sibuya's solution is subdominant in S_0. For fixed z, it is an entire function of λ of order

$$\rho = 5/6.$$

All these results belong to Y. Sibuya.

It follows from the definition that solutions y_k are subdominant in S_k. Solutions y_0, y_1 are linearly independent because no solution can be subdominant in adjacent sectors. Every three solutions must be linearly dependent, so

$$y_{-1} = C(\lambda)y_0 + \tilde{C}(\lambda)y_1.$$

In the sector S_0, all three have asymptotics which follow from (3). Comparing the asymptotics of y_1 and y_{-1} on the positive ray (where y_0 tends to zero and does not contribute) we conclude that $\tilde{C} = -\omega$, independent of λ.

Thus

$$y_{-1} = C(\lambda)y_0 - \omega y_1. \tag{4}$$

Function $C(\lambda)$ is called the Stokes multiplier. It is an entire function. Indeed, we differentiate (4) with respect to z and obtain

$$y'_{-1} = C(\lambda)y'_0 - \omega y'_1, \tag{5}$$

and solve (4), (5) for C using Cramer's rule. We obtain

$$C(\lambda) = W_{-1,1}/W_{0,1},$$

where $W_{i,j}$ is the Wronskian of y_i, y_j. Now y_0 and y_1 are linearly independent for each λ, therefore $W_{0,1}$ is free of zeros, and C is an entire function. Equations (2), (4) give an explicit expression of C in terms of Sibuya's solution y_0, and also show that C is of order at most $5/6 < 1$.

More precisely, let us consider the entire function $f(\lambda) = y_0(0, \lambda)$. Zeros of f are exactly the eigenvalues of the equation (1) with the boundary conditions $y(0) = y(+\infty) = 0$. As this eigenvalue problem is self-adjoint and the potential z^3 is positive on the positive ray, we conclude that all these zeros are negative. Thus

$$f(\lambda) = c \prod_{k=1}^{\infty} \left(1 + \frac{\lambda}{\lambda_k}\right), \tag{6}$$

where $c \neq 0$ is a real constant.

Lemma. Dorey, Dunning, Tateo (2001), K. Shin (2002) *All zeros of the Stokes multiplier C are positive.*

Proof of the Lemma. Let λ be a zero of C. Plugging λ to (4) with $z = 0$, and using the definition of y_k we obtain

$$|f(\omega^2\lambda)| = |f(\omega^{-2}\lambda)|. \tag{7}$$

For any fixed $r > 0$, the function $\theta \mapsto |f(re^{i\theta})|$ is even and strictly increasing on $[0, \pi]$ which is immediately seen from (6), where every factor enjoys these properties. Thus λ in (7) must be real.

Now we prove that C has no negative zeros. Let λ be a zero of C. We know that it is real and that for this λ we have $y_{-1} = -\omega y_1$ in view of (4). This means that there is a solution y subdominant in two sectors S_1 and S_{-1}. Let $y^*(z) = \overline{y(\overline{z})}$; this solves the same equation and is subdominant in the same sectors, from which we conclude that $y^* = cy$. Plugging any real value for z, we obtain $|c| = 1$. Then $w = y\sqrt{c}$ is a real solution, subdominant in S_1. Now we substitute $h(t) = w(te^{i\alpha})$ with $\alpha = 2\pi/5$ into (1), multiply on $\overline{h(t)}$ and integrate from 0 to ∞. We obtain after an integration by parts,

$$-w'(0)\overline{w(0)} - e^{-i\alpha} \int_0^\infty |h'|^2 dt - \int_0^\infty t^3 e^{4i\alpha} |h|^2 dt = \lambda e^{i\alpha} \int_0^\infty |h|^2 dt.$$

Taking the imaginary part and using the fact that w is real on the real line, so $w'(0)\overline{w(0)}$ is real, we obtain

$$\sin\alpha \int_0^\infty |h'|^2 dt - \sin(4\alpha) \int_0^\infty t^3 |h|^2 dt = \lambda \sin\alpha \int_0^\infty |h|^2 dt.$$

As $\sin\alpha > 0$ while $\sin 4\alpha < 0$, we conclude that the left hand side is positive, thus $\lambda > 0$. This proves the lemma.

Substituting to (4) $(z, \lambda) \mapsto (\omega^{-1} z, \omega^2 \lambda)$, we obtain

$$y_0 = C(\omega^2\lambda) y_1 - \omega y_2. \tag{8}$$

Using this to eliminate y_0 from (4), we obtain

$$y_{-1} = \Big(C(\lambda) C(\omega^2\lambda) - \omega \Big) y_1 - C(\lambda) \omega y_2. \tag{9}$$

Notice that according to our definition, $y_k = y_j$ when $k \equiv j \pmod 5$, in particular, $y_{-1} = y_4$. Substituting to (4) $(z, \lambda) \mapsto (\omega^{-3} z, \omega\lambda)$ we obtain a relation

$$y_2 = C(\omega\lambda) y_3 - \omega y_{-1} \tag{10}$$

We conclude from (9) and (10) that

$$C(\lambda) C(\omega^2\lambda) - \omega \quad \text{and} \quad C(\omega\lambda)$$

have the same zeros: indeed these zeros are exactly those λ for which y_2 and y_{-1} are linearly dependent. As C has genus 0, it is determined by its zeros up to a constant factor. To find this factor, we plug $(z, \lambda) = (0, 0)$ to (4) and find

$$C(0) = 1 + \omega.$$

Thus

$$C(\omega\lambda)C(\omega^{-1}\lambda) - \omega = bC(\lambda), \tag{11}$$

where $b = (1 + \omega + \omega^2)/(1 + \omega)$. Now the function $C(\lambda)/\sqrt{\omega}$ has positive zeros and its 1-points lie on the rays $\{t\omega : t > 0\}$ and $\{t/\omega : t > 0\}$. This proves the theorem for $\alpha = 2\pi/5$.

To extend this to other angles α, the natural idea is to consider the differential equation

$$-y'' + (z^m + \lambda)y = 0 \tag{12}$$

with real $m > 2$. The relation between m and α is

$$\alpha = \frac{2\pi}{m + 2}.$$

Such equations were studied by the physicists Bender and Boettcher, who computed few first zeros of the relevant Stokes multipliers. Their computation indicates that these zeros are all real when $m \geq 4$. This was known for integer m and was rigorously proved for all $m > 4$ in my recent preprint thus giving the range $\alpha \in (0, \pi/3]$ in Theorem 2. However for $2 < m < 4$, except $m = 3$, which corresponds to $\alpha \in (\pi/3, \pi/2)$ except $2\pi/5$, the computation of Bender and Boettcher shows that only finitely many zeros are real, though this fact is unproved. This shows that our method will not give the desired result for the remaining range $(\pi/3, \pi/2)$ of α.