

## Holomorphic curves with few inflection points

Consider a vector of linearly independent entire functions  $f = (f_0, \dots, f_n)$ , whose coordinates have no common zeros. The *characteristic* is defined by

$$T(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log \|f(re^{i\theta})\| d\theta - \log \|f(0)\|,$$

and the *lower order* is

$$\lambda = \liminf_{r \rightarrow \infty} \frac{\log T(r, f)}{\log r}. \quad (1)$$

We also define

$$N_1(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |W(re^{i\theta})| d\theta,$$

where  $W = W(f_0, \dots, f_n)$  is the Wronskian determinant.  $N_1$  is called the ‘ramification term’ in the Second Main Theorem of Cartan. Actually this is the averaged counting function of inflection points of the holomorphic curve  $(f_0, \dots, f_n)$  in projective space.

**Conjecture.** Suppose that  $\lambda < \infty$  and  $N_1(r) = o(T(r))$ . Then  $\lambda$  is rational and the limit in (1) exists.

This is known to be the case when  $n = 1$  [1].

Here is the motivation for the conjecture (the argument is due to V. P. Petrenko). Suppose that  $\lambda < \infty$ , and  $N_1(r) \equiv 0$ . This means that the Wronskian has no zeros, so  $f_0, \dots, f_n$  make a fundamental system of solutions to a differential equation

$$w^{(n+1)} + P_1 w^{(n)} + \dots + P_0 w = 0,$$

where  $P_j$  are entire functions. By a theorem of M. Frei [2], our condition that  $\lambda < \infty$  implies that  $P_j$  are polynomials. In this case, the conclusion of the conjecture is well known.

## References

- [1] A. Eremenko, Meromorphic functions with small ramification, *Indiana Univ. Math. J.*, 42 (1993), 1193-1218.
- [2] M. Frei, Sur l'ordre des solutions entières d'une équation différentielle linéaire, *C. r.*, 236 (1953), 38-40.