1.1 Exact equations. Every first-order differential equation

$$y' = f(x, y)$$

can be rewritten in a *symmetric form*

$$M(x, y)dx + N(x, y)dy = 0. \tag{1}$$

The equation (1) is called *exact* if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \tag{2}$$

If this condition is satisfied, one can find a function $F(x, y)$ such that

$$dF(x, y) = M(x, y)dx + N(x, y)dy, \tag{3}$$

which means that the family of implicitly defined functions

$$F(x, y) = c \tag{4}$$

is a general solution of (1).

1.2 An example of exact equation is

$$e^y dx + (xe^y + 2y)dy = 0, \quad \text{whose general solution is} \quad xe^y + y^2 = c.$$

Every separable equation can be written in a symmetric form

$$f(x)dx + g(y)dy = 0,$$

which is also exact.

1.3 How to solve exact equations. (3) is the same as

$$\frac{\partial F}{\partial x} = M \quad \text{and} \quad \frac{\partial F}{\partial y} = N. \tag{5}$$

Thus to find F we have to solve (5). Integrating the first equation in (5) with respect to x we obtain

$$F(x, y) = \int M(x, y)dx + g(y), \tag{6}$$

where g is still unknown. To find g we differentiate (6) with respect to y and plug to the second equation in (5). We obtain:

$$\frac{\partial}{\partial y} \int M(x, y)dx + g'(y) = N(x, y),$$

which implies

$$g(y) = \int \left\{ N(x,y) - \frac{\partial}{\partial y} \int M(x,y)dx \right\} dy. \tag{7}$$

It remains to plug this expression to (6).

1.4 For example, to solve $e^y dx + (xe^y + 2y)dy = 0$ (which is exact), we write

$$\frac{\partial F}{\partial x} = e^y \quad \text{and} \quad \frac{\partial F}{\partial y} = xe^y + 2y.$$

Integrating the first equation with respect to x gives

$$F = \int e^y dx + g(y) = xe^y + g(y),$$

so

$$\frac{\partial F}{\partial y} = xe^y + g'(y).$$

Substituting this to the second equation, we obtain $g'(y) = 2y$, so $g(y) = y^2 + \text{const}$, and $F = xe^y + y^2 + \text{const}$. Thus the general solution is given by

$$xe^y + y^2 = c.$$