2.1 Integrating factors. If the differential equation

$$M(x,y)dx + N(x,y)dy = 0 \tag{1}$$

is not exact, one can try to find a function $I(x,y)$, such that the equation multiplied by $I(x,y)$ is exact. Then the general solution of this exact equation will be also the general solution of the original equation.

2.2 For example, the separable equation $ydx + 2xdy = 0$ is not exact, but after multiplication by $1/(xy)$ it becomes $x^{-1}dx + 2y^{-1}dy = 0$, which is exact.

2.3 Finding an integrating factor may not be easy (no general algorithm exists). We list several cases when it is known how to find it. Let us write the condition that I is an integrating factor for the equation (1). This means that the equation $IMdx + INdy = 0$ is exact, so

$$\frac{\partial(IM)}{\partial y} = \frac{\partial(IN)}{\partial x},$$

or

$$I_xN - I_yM = I(M_y - N_x). \tag{2}$$

2.4 An integrating factor independent of x**.** If an integrating factor I, which does not depend on x exists, it satisfies by (2)

$$-I_yM = I(M_y - N_x).$$

Dividing by I we conclude

$$\frac{I_y}{I} = N_x - M_y. \tag{3}$$

The left hand side is independent of x, so the right hand side should be also independent of x. If this is the case, I can be found by integration with respect to y:

$$\int \frac{I_y}{I} dy = \int (N_x - M_y)dy.$$

The left hand side is equal to $\log I + \text{const}$, so

$$I = \exp \int (N_x - M_y)dy. \tag{4}$$

Thus an integrating factor independent of x exists if and only if $N_x - M_y$ is independent of x, and in this case it is given by the expression (4).

The case when an integrating factor independent of y exists, can be treated similarly.

2.5 As an example we consider a linear differential equation:

$$y' + p(x)y + q(x) = 0. \tag{5}$$

When written in symmetric form, (5) becomes

$$dy + (p(x)y + q(x))dx = 0,$$

so $M(x, y) = p(x)y+q(x)$ and $N(x, y) = 1$. Thus $N_x - M_y = p(x)$ is independent of y and so

$$I = \exp \int p(x)dx \tag{6}$$

is an integrating factor. Thus the differential equation

$$e^{\int p(x)dx}dy + e^{\int p(x)dx}(p(x)y + q(x))dx = 0$$

is exact. Its general solution is

$$y = e^{-\int p(x)dx} \left\{ - \int q(x)e^{\int p(x)dx}dx \right\}.$$

Notice that there is one arbitrary constant involved in this last formula.