Solution of additional Problem 4

It is easy to see that there is one root in the unit disc, because $4|z| = 4 > 3 \geq |z^4 + z^3 + 1|$ for $|z| = 1$.

Now let us show that there are 4 roots in $|z| < 2$. If we split our function like this

$$z^4 + z^3 - 4z + 1 = (z^4 + 1) + (z^3 - 4z),$$

it will be enough to prove that

$$|z^3 - 4z| < |z^4 + 1| \text{ for } |z| = 2. \quad (1)$$

This is equivalent to the following chain of inequalities:

$$2|z^2 - 4| < |z^4 + 1|, \quad |z| = 2,$$

$$2|w - 4| < |w^2 + 1|, \quad |w| = 4,$$

$$4(w - 4)(\bar{w} - 4) < (w^2 + 1)(\bar{w}^2 + 1) \quad |w| = 4;$$

$$4(32 - 32 \cos t) < 257 + 64 \cos(2t),$$

where we put $w = 4 \exp(it)$. And this is the same as

$$128 \cos^2 t + 128 \cos t + 65 > 0.$$  

Denoting $\cos t = x$ and finding the minimum of the LHS, we find that this minimum is 1 (it occurs for $x = -1/2$). Thus the last inequality is true and this proves (1).