Consider a general second order ODE of the form

$$y'' = f(x, y, y'). \tag{1}$$

There are some cases when it can be reduced to a first order equation.

1. Assume that the variable x is missing in the RHS, that is our equation has the form

$$y'' = f(y, y').$$

Then we can introduce a new variable w by the equation $y' = w(y)$. Differentiating this with respect to the independent variable x yields $y'' = w'(y)y' = w'(y)w(y)$. Substituting this to the equation, we obtain

$$w'w = f(y, w),$$

which can be considered as a *first-order equation with y as an independent variable, and w unknown function.* Assume that we can solve this first order equation, and $w(y, C_1)$ is an explicit solution. Then we return to our change of variable

$$y' = w(y, C_1),$$

and solve this last equation (it is separable). We will obtain

$$\int \frac{dy}{w(y, C_1)} = \int dx = x + C_2,$$

so the general solution involves two arbitrary constants, as one expects.

For example, to solve

$$yy'' + (y')^2 = 0,$$

we set $y' = w(y),\ y'' = w'w$, and obtain

$$yw'w + w^2 = 0, \quad \text{or} \quad yw' + w = 0,$$

where y is considered an independent variable here. The last equation is separable, and its general solution is $w(y) = C_1/y$. Returning to the original variable, we have to solve $y' = C_1/y$, which is also separable and has general solution $y^2/2 = C_1x + C_2$.

2. Assume that the function y itself is missing in the RHS, that is

$$y'' = f(x, y').$$

Then we make a change of variable $y' = w$, so $y'' = w'$ and the problem is reduced to the first order equation

$$w' = f(x, w).$$

After it is solved we will have to integrate one time to get y.

3. You may expect now that if y' is missing in the RHS of (1), one can also make a smart change of variable to reduce the order of the equation. This is not the case. A simple equation of this sort (with y' missing) is

$$y'' + xy = 0,$$

which is known as the Airy equation. This is the simplest equation about which *it is proved that none of its solutions is an elementary function*, except the zero solution, of course. Nevertheless all solutions of this equation are very nice functions, defined for all real (and even all complex) values of x. They are called Airy functions, and very much is known about them. They are studied in more advanced courses of differential equations.

3*. (This case is rarely included in undergraduate courses, and it will not be required in exams). Consider a homogeneous equation with respect to y'', y' and y, that is

$$F(y'', y', y, x) = 0, \tag{2}$$

where F is a homogeneous polynomial of some degree m with respect to y'', y', y, whose coefficients may depend on x. For example,

$$yy'' + x(y')^2 = 0 \tag{3}$$

is homogeneous of degree 2 (x is not counted here!) The trick is to divide equation (2) by y^m, where m is the degree of homogeneity. Then only combinations of the form y'/y and y''/y will remain in the equation. That is the equation will be of the form

$$G(y'/y, y''/y, x) = 0. \tag{4}$$

Now we introduce a new (dependent) variable $w = y'/y$. To express y''/y in terms of w, we differentiate:

$$w' = \frac{y''y - (y')^2}{y^2} = \frac{y''}{y} - \left(\frac{y'}{y}\right)^2 = \frac{y''}{y} - w^2.$$

Thus $y''/y = w' + w^2$. When we substitute this to (4), the result will be

$$G(w' + w^2, w, x) = 0,$$

which is of first order. Assuming that we can solve it and find $w(x, C_1)$, it remains to solve $y'/y = w(x, C_1)$, which is separable. Thus the final answer will be $y = C_2 \exp \int w(x, C_1) dx$.

For example, to solve (3) we first divide it by y^2 and obtain

$$\frac{y''}{y} + x \left(\frac{y'}{y} \right)^2 = 0.$$

Setting $w = y'/y$ and differentiating this, we obtain as before, $y''/y = w' + w^2$. Thus

$$w' + w^2 + xw^2 = 0, \quad \text{or} \quad -\frac{w'}{w^2} = 1 + x.$$

The last equation is separable, and its general solution is $w = (x + x^2/2 + C_1)^{-1}$. Returning to the original variable y, we have

$$\frac{y'}{y} = \frac{2}{x^2 + 2x + C_2} = \frac{2}{(x+1)^2 + C_3}.$$

Integration gives

$$y = \exp \int \frac{2dx}{(x+1)^2 + C_3}.$$

Evaluation of the integral is not hard here, but we have to consider two cases, depending on the sign of C_3, or to use complex numbers.