Space travel

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1. A stone thrown vertically upwards will reach some maximal height and return. If the initial speed increases this maximal height increases. Does there exist an initial speed for which the stone will never return? This is called the *escape speed*.

We neglect resistance and only take the gravity force into account. Let x be the distance from the center of the Earth, v - velocity, (speed), and m -mass of the body. The equation of motion is

$$m\frac{d^2}{dt^2}x = -m\gamma x^{-2},$$

where $\gamma > 0$ is a constant. This is Newton's second law, combined with the gravitation law. Dividing by m we obtain

$$\frac{d^2x}{dt^2} = -\gamma x^{-2}.$$

This is a non-linear second order equation. The independent variable is time. This equation simplifies if we introduce the speed

$$\frac{dx}{dt} = v(x)$$

as a function of distance, rather than time. Then differentiating with respect to time we obtain

$$\frac{d^2x}{dt^2} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

and our equation becomes

$$vdv = -\gamma x^{-2}dx.$$

This is a separable first order equation. Integrating it we obtain

$$\frac{1}{2}v^2 = \gamma x^{-1} + C.$$

Notice that this is just the energy conservation law: when multiplied back by m, the LHS is the kinetic energy and the RHS is the negative of the potential energy in the gravity field.

Suppose that the motion starts on the Earth surface, $x(0) = R$ with initial speed $v(R) = v_0$. Then

$$C = \frac{1}{2}v_0^2 - \gamma/R.$$

If v_0 is the escape speed, that is minimal speed for which the projectile will not return, then we have $v(t) \to 0$, and $x(t) \to \infty$ as $t \to \infty$, which gives $C = 0$, and we can find v_0:

$$v_0 = \sqrt{2\gamma/R} = \sqrt{2Rg},$$

where the second equation holds because $g = \gamma R^{-2}$, acceleration of gravity on the Earth surface. Substituting $g = 9.8\,m/sec$ and $R = 6,400,000\,m$ we obtain the escape speed for the Earth, $v_0 \approx 11,200\,m/sec.$

This is a pretty high speed by usual standards.

In his famous novel, Jules Verne proposed to build an enormous gun shooting the projectile vertically. However the speed of a projectile shot from a gun cannot exceed the expansion speed of the propellant (powder) and the maximal speed achieved with chemical propellants is substantially smaller than the escape speed. So one cannot achieve this speed with a gun.

2. Rockets. A rocket moves because it throws back the jet of burned fuel. Let c be the speed of this jet (with respect to the rocket, of course). Now, in the reference frame where the rocket and fuel are initially at rest, the momentum must be preserved. When the burned fuel is ejected on some direction, the rocket must move in the opposite direction, so that the total momentum remains zero. We neglect here all external forces.

Let M be the initial mass of the rocket, with fuel, and m the mass of empty rocket. Let $y(t)$ be the mass of fuel spent up to time t. Then the mass of the rocket at time t will be $M - y(t)$. If $v(t)$ is the speed of the rocket, then momentum is $v(t)(M - y(t))$. The rate of change of this momentum plus the rate of change of the momentum of the jet must be zero:

$$\frac{d}{dt}\left(v(t)(M - y(t))\right) + (v - c)y' = 0,$$

because the velocity of the jet with respect to the initial frame of reference is $v - c$ and y' is the rate of mass loss of fuel (which is converted by the engine into the jet). Simplifying we obtain

$$(M - y)dv = cdy,$$

or

$$dv = \frac{cdy}{M - y},$$

which is a separable equation. Solving it we obtain

$$v(y) = -c\log(M - y) + C_0.$$

In the beginning, $y = 0$ and $v = 0$, so $C_0 = c\log M$, and the final result is

$$v(y) = c\log\frac{M}{M - y}.$$

When all fuel is spent, $y = M - m$ and we obtain that the maximal speed is

$$v_{\max} = c\log\frac{M}{m}.$$

It is remarkable and somewhat counter-intuitive that speeds higher than c can be achieved. Indeed, when $v > c$ the material of the jet moves in the *same direction as the rocket* from the point of view of the observer for which the rocket was initially at rest !

In principle, with given c, arbitrarily high speed can be achieved, but this is not so from the practical point of view. The reason is that $\log(M/m)$ increases with M/m *very slowly*. And of course there are engineering limitations on the ratio M/m: the fuel must be stored somewhere, and you cannot indefinitely increase the ratio of the mass of fuel to the mass of the storage tank.

A partial remedy consists in constructing multi-stage rockets: when most of the fuel is spent, the fuel tank (and the engine) of the first stage are dropped, and the much smaller second stage engine starts working. When its tanks are empty it is also dropped, and the third stage begins to work.

For example, Saturn 5 rocket, which was used to bring Apollo spacecraft to the Moon has three stages with the following approximate characteristics:

First stage with $M = 2.29$ and $m = 0.13$ (thousand tons),

Second stage with $M = 496$ and $m = 40$ (tons), and

Third stage with $M = 123$ and $m = 13$ tons.

So for the whole system $M = 2.9$ thousand tons, and useful load about 13 tons. Of course not all the difference $M - m$ was fuel, but some of it was the material of the first and second stages.