The semigroup approach
for measure-valued branching processes
and a non-linear Dirichlet problem

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$E$ : a Lusin topological space with Borel $\sigma$-algebra $\mathcal{B}$

$\mathcal{L}$ : the generator of a right Markov process
$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$ with state space $E$:

- $(\Omega, \mathcal{F})$ is a measurable space, $P^x$ is a probability measure on $(\Omega, \mathcal{F})$ for every $x \in E$

- $(\mathcal{F}_t)_{t \geq 0}$: filtration on $\Omega$

- The mapping $[0, \infty) \times \Omega \ni t \mapsto X_t(\omega) \in E$ is $\mathcal{B}([0, \infty)) \times \mathcal{F}$-measurable

- $X_t$ is $\mathcal{F}_t/\mathcal{B}$-measurable for all $t$
– There exists a semigroup of kernels \((P_t)_{t \geq 0}\) on \((E, \mathcal{B})\), called **transition function** of \(X\), such that for all \(t \geq 0, x \in E\) and \(A \in \mathcal{B}\) one has

\[
P^x(X_t \in A) = P_t(x, A)
\]

[ If \(f \in p\mathcal{B}\) then \(E^x(f \circ X_t) = P_t f(x)\) ]

– For each \(\omega \in \Omega\) the mapping

\[
[0, \infty) \ni t \mapsto X_t(\omega) \in E
\]

is right continuous
Let \( \mu \) be a finite measure on \( E \). The right process \( X \) has \textbf{càdlàg trajectories} \( \mu \)-a.e. if it possesses left limits in \( E \) \( \mu \)-a.e. on \([0, \zeta)\); \( \zeta \) is the life time of \( X \).
The resolvent of the process $X$

\[ \mathcal{U} = (U_q)_{q > 0}, \]

\[ U_q f(x) := E^x \int_0^\infty e^{-qt} f \circ X_t dt = \int_0^\infty e^{-qt} P_t f(x) dt, \quad x \in E, \ q > 0, \ f \in p\beta \]

$\mathcal{L}$ is the infinitesimal generator of $(U_q)_{q > 0}$, $[U_q = (q - \mathcal{L})^{-1}]$
The following properties are equivalent for a function \( v : E \rightarrow \overline{\mathbb{R}}_+ \):

\( (i) \) \( v \) is \((\mathcal{L} - q)\)-superharmonic

\( (ii) \) There exists a sequence \((f_n)_n\) of positive, bounded, Borel measurable functions on \( E \) such that \( U_q f_n \uparrow v \)

\( (iii) \) \( \alpha U_{q+\alpha} v \leq v \) for all \( \alpha > 0 \) and \( \alpha U_{q+\alpha} v \uparrow v \)

\( S(\mathcal{L} - q) \) : the set of all \((\mathcal{L} - q)\)-superharmonic functions
If $M \in \mathcal{B}$, $q > 0$, and $u \in S(\mathcal{L} - q)$ then the **reduced function** of $u$ on $M$ (with respect to $\mathcal{L} - q$) is the function $R_d^M u$ defined by

$$R_d^M u := \inf \{ v \in S(\mathcal{L} - q) : v \geq u \text{ on } M \}. $$

- The reduced function $R_d^M u$ is universally $\mathcal{B}$-measurable.

- Let $p := U_q 1$. The functional $M \mapsto c_\mu(M)$, $M \subset E$, defined by

$$c_\mu(M) := \inf \{ \int_E R_q^G p \, d\mu : G \text{ open }, M \subset G \}$$

is a Choquet capacity on $E$.

- $R_d^M f(x) = E^x(f(X_{D_M}))$ [Hunt’s Theorem]
The capacity $c_\mu$ is **tight** provided that there exists an increasing sequence $(K_n)_n$ of compact sets such that

$$\inf_n c_\mu(E \setminus K_n) = 0$$

or equivalently,

$$P^\mu(\lim_n D_{E \setminus K_n} < \zeta) = 0.$$
Sufficient conditions for tightness

• If $X$ has càdlàg trajectories $P^\mu$-a.e. then the capacity $c_\mu$ is tight.


• If there exists a dual process of $X$ with respect to a duality measure $m$ and $\mu \ll m$, then the capacity $c_\mu$ is tight.

(weak duality $\implies$ tightness)
A $(\mathcal{L} - q)$-superharmonic function $v$ is called **compact Lyapunov function** provided that it is finite $\mu$-a.e. and the set $[v \leq \alpha]$ is relatively compact for all $\alpha > 0$.

**Proposition**

Assume that $(P_t)_{t \geq 0}$ is Markovian, i.e., $P_t 1 = 1$ for all $t > 0$. Then the following assertions are equivalent.

(a) The capacity $c_\mu$ is tight.

(b) There exists a compact Lyapunov function.


Let $\mathcal{U} = (U_\alpha)_{\alpha > 0}$ be a sub-Markovian resolvent of kernels on a Lusin topological space $(E, \mathcal{T})$ and assume that:

(A) $\sigma(S(L - q)) = \mathcal{B}$ and all the points of $E$ are non-branch points with respect to $\mathcal{U}_q$.

(B) For each $x \in E$ there exists a function $u \in S(L - q)$ such that $u(x) < \infty$ and the set $[u \leq k]$ is relatively compact for all $k$ ($u$ is a compact Lyapunov function).

(C) There exists a countable family $\mathcal{F}$ of bounded $\mathcal{B}$-measurable functions, $\mathcal{F} \subset [bS(L - q)]$, such that $\mathcal{T}$ is generated by $\mathcal{F}$.

(D) ("Uniqueness of charges" type property) If $\xi, \eta$ are two $\mathcal{U}_q$-excessive measures such that $L_q(\xi, \varphi) = L_q(\eta, \varphi)$ for all $\varphi \in \mathcal{F}$, then $\xi = \eta$.

Then there exists a càdlàg (right Markov) process with state space $E$, having $\mathcal{U}$ as associated resolvent.

Applications

A1. Explicit construction of compact Lyapunov functions for Lévy processes on Hilbert spaces
[L.B., A. Cornea, & M. Röckner, JFA 2011]

A2. Singular SDE on Hilbert spaces

Construction of measure-valued branching processes associated to some nonlinear PDEs

A3. Discrete branching processes

A4. Continuous branching processes
• M. Nagasawa ([Sémin. de probab. (Strasbourg) 10, 1976, pp. 184-193]) related this nonlinear problem to a branching Markov process.
• P.J. Fitzsimmons: Construction and regularity of measure-valued Markov branching processes, Israel J. Math. 64, 337-361, 1988
• Li, Zenghu Measure-Valued Branching Markov Processes. (Probab. Appl.), Springer 2011
$M(E)$: the space of all positive finite measures on $(E, \mathcal{B})$ endowed with the weak topology.

For a function $f \in b \mathcal{B}$ consider the mappings

$l_f : M(E) \rightarrow \mathbb{R},$

$$l_f(\mu) := \langle \mu, f \rangle := \int f d\mu, \ \mu \in M(E),$$

$e_f : M(E) \rightarrow [0, 1]$

$$e_f := \exp(-l_f).$$

$\mathcal{M}(E)$: the $\sigma$-algebra on $M(E)$ generated by $\{l_f | f \in b \mathcal{B}\}$, the Borel $\sigma$-algebra on $M(E)$
The space of finite configurations of $E$

$S$: the set of all positive measures $\mu$ on $E$ which are finite sums of Dirac measures, $\mu = \sum_{k=1}^{m} \delta_{x_k}$, where $x_1, \ldots, x_m \in E$.

$S$ is identified with the direct sum of all symmetric $m$-th powers $E^{(m)}$ of $E$, hence

$$S = \bigoplus_{m \geq 1} E^{(m)},$$

and it is equipped with the canonical topological structure.

$\mathcal{B}(S)$: the Borel $\sigma$-algebra on $S$. 

Multiplicative functions

- Let $\varphi \in pB$, $\varphi \leq 1$. Consider the function $\hat{\varphi} : S \rightarrow \mathbb{R}$, called \textbf{multiplicative}, defined as

  \[ \hat{\varphi}(x) := \varphi(x_1) \cdot \ldots \cdot \varphi(x_m) \text{ for } x = (x_1, \ldots, x_m) \in E^{(m)}. \]

- A multiplicative function $\hat{\varphi}$ is the restriction to $S$ of an exponential function on $M(E)$,

  \[ \hat{\varphi} = e^{-\ln \varphi}. \]
Let $p_1, p_2$ be two finite measures on $M(E)$.

- **The convolution $p_1 \ast p_2$:** the finite measure on $M(E)$ defined for every bounded Borel function $F$ on $M(E)$ by

  $$
  \int p_1 \ast p_2 (d\nu) F(\nu) := \int p_1 (d\nu_1) \int p_2 (d\nu_2) F(\nu_1 + \nu_2).
  $$

- If $p_1$ and $p_2$ are concentrated on $S$ then $p_1 \ast p_2$ has the same property and

  $$
  p_1 \ast p_2 (\hat{\varphi}) = p_1 (\hat{\varphi}) p_2 (\hat{\varphi}).
  $$

- **Branching kernel:** a kernel $K$ on $M(E)$ such that for all $\mu, \nu \in M(E)$ we have

  $$
  K_{\mu + \nu} = K_\mu \ast K_\nu.
  $$
A Markov process $\bar{X}$ with state space $M(E)$ is called a branching process provided that for all $\mu_1, \mu_2 \in M(E)$, the process $X^{\mu_1 + \mu_2}$ starting from $\mu_1 + \mu_2$ and the sum $X^{\mu_1} + X^{\mu_2}$ are equal in distributions, i.e., for all $t \geq 0$ and $F \in bpM(E)$ we have

$$\int F(\bar{X}_t(\omega))\overline{P}^{\mu_1 + \mu_2}(d\omega) = \int \int (F(\bar{X}_t(\omega) + \bar{X}_t(\omega'))\overline{P}^{\mu_1}(d\omega)\overline{P}^{\mu_2}(d\omega')$$

$\bar{X}$ is a branching process $\iff \overline{P}_t$ is a branching kernel for all $t$. 
• An initial particle starts at a point of $E$ and moves according to a base process $X$ (with state space $E$) until a random time (defined by killing $X$) when it splits into a random number $m$ of new particles, its direct descendants, placed in $E$.

• Each direct descendant moves on according to the $m$ independent copies of $X$ an so on.
Proposition

(i) For every sub-Markovian kernel $B : pB(S) \rightarrow pB$ there exists a branching kernel $\hat{B}$ on $(S, B(S))$ such that for every $\varphi \in pB$, $\varphi \leq 1$, we have

$$\hat{B}\varphi = \hat{B}\varphi.$$ 

(ii) Conversely, if $H$ is a branching kernel on $(S, B(S))$ then there exists a unique sub-Markovian kernel $B : pB(S) \rightarrow pB$ such that $H = \hat{B}$. 
Let \( q_k \in \mathcal{pB} \) for all \( k \geq 1 \), satisfying \( \sum_{k \geq 1} q_k \leq 1 \).

Consider the kernel \( \hat{B} \), where \( B : \mathcal{pB}(S) \to \mathcal{pB} \) is defined as

\[
(*) \quad Bg(x) := \sum_{k \geq 1} q_k(x)g_k(x, \ldots, x), \quad g \in b\mathcal{pB}(S), \quad x \in E
\]

with \( g_k := g|_{E(k)} \).

In particular, for all \( \varphi \in \mathcal{pB} \), \( \varphi \leq 1 \), we have

\[
B\hat{\varphi} = \sum_{k \geq 1} q_k\varphi^k.
\]
Branching processes on the finite configurations

- **Base process**: $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, \theta_t, P^x)$, a right (Markov) process with state space $E$

- **Branching kernel**: a sub-Markovian kernel $B : bp\beta(S) \rightarrow bp\beta$ such that

  $\sup_{x \in E} B l_1(x) < \infty$.

If $B$ is given by $(\ast)$ then the above condition is equivalent with

$\sup_{x \in E} \sum_{k \geq 1} k q_k(x) < \infty$.

- **Killing kernel**: $c \in bp\beta$
Theorem

There exists a branching semigroup $(\hat{H}_t)_{t \geq 0}$ on $S$, having the following property: If $u \in pB$, $u \leq 1$, is such that $\hat{u}$ is an invariant function with respect to this semigroup, then $u$ belongs to the domain of $\mathcal{L}$ (the infinitesimal generator of the base process $X$) and

$$(\mathcal{L} - c)u + cB\hat{u} = 0.$$  

In particular, if $B$ is given by ($\ast$), then $u$ is the solution of the nonlinear equation

$$(\mathcal{L} - c)u + c \sum_{k \geq 1} q_k u^k = 0.$$
Absolutely monotonic map, [Silverstein, 68]

$B_1 :=$ the set of all functions $\varphi \in pB$ such that $\varphi \leq 1$.

**Absolutely monotonic map:** $H : B_1 \rightarrow B_1$ such that that there exists a sub-Markovian kernel $H : bpB(S) \rightarrow bpB$ with $H\varphi = H\hat{\varphi}$ for all $\varphi \in B_1$.

- A map $H : B_1 \rightarrow B_1$ is absolutely monotonic if and only if there exists a branching kernel $\hat{H}$ on $S$ such that $\hat{H}\hat{\varphi} = \hat{H}\varphi$ for all $\varphi \in B_1$. 
The existence of the branching semigroup
(sket of the proof)

Proposition

For every $\varphi \in \mathcal{B}_1$ the equation

$$w_t(x) = T_t \varphi(x) + \int_0^t T_s(c B \hat{w}_{t-s})(x) \, ds, \quad t \geq 0, \quad x \in E$$

has a unique solution $(t, x) \mapsto \mathcal{H}_t \varphi(x)$ jointly measurable in $(t, x)$, such that $\mathcal{H}_t \varphi \in \mathcal{B}_1$ and the following assertions hold.

(i) The mapping $\varphi \mapsto \mathcal{H}_t \varphi$ is absolutely monotonic for all $t \geq 0$.

(ii) The family $(\mathcal{H}_t)_{t \geq 0}$ is a nonlinear semigroup of operators on $\mathcal{B}_1$. 
- Since $H_t$, $t > 0$, is absolutely monotonic, there exists a branching kernel $\hat{H}_t$ on $(S, \mathcal{B}(S))$, $t \geq 0$, such that for all $\varphi \in \mathcal{B}_1$

$$H_t\varphi = \hat{H}_t\varphi|_E.$$ 

- There exists a family $(V_t)_{t \geq 0}$, a (nonlinear) semigroup on $bp\mathcal{B}$, such that

$$\hat{H}_t(e_f) = e_{V_t f}, \quad f \in bp\mathcal{B}.$$ 

More precisely, $V_t f := -\ln H_t(\exp(-f))$. 

If $X$ is a Hunt process with state space $E$, then $(\hat{H}_t)_{t \geq 0}$ is the transition function of a càdlàg branching process with state space $S$.

[L.B. & O. Lupascu, in preparation]
Remark

(i) If $u \in pB$ is an excessive function with respect to the base process $X$, then $l_u$ is excessive with respect to the branching semigroup $(\hat{H}_t)_{t \geq 0}$.

(ii) If $u$ is a compact Lyapunov function for the base process $X$, then $l_u$ is a compact Lyapunov function for the branching semigroup $(\hat{H}_t)_{t \geq 0}$. In particular, condition (B) holds for the branching semigroup.
A nonlinear Dirichlet problem

- $D$: a bounded open subset of $\mathbb{R}^d$
- $(q_n)_{n \geq 2} \subset C^+_b(D)$ such that $\sum_{n \geq 2} q_n \leq 1$, $c \in C^+_b(D)$
- $\varphi$: a bounded measurable function on the boundary $\partial D$ of $D$
- Let $\mathcal{L}$ be the infinitesimal generator of a Feller semigroup on $D$ and consider the following nonlinear Dirichlet problem:

$$\begin{cases}
\mathcal{L}u + c(\sum_{n \geq 2} q_n u^n - u) = 0 \text{ on } D, \\
u = \varphi \text{ on } \partial D.
\end{cases}$$
• $Y$ is a transient, path continuous right Markov process with state space a Lusin topological space $F$.

• $D$ is an open subset of $F$ and suppose that every point of the boundary $\partial D$ of $D$ is regular, i.e. $P^y(T_{F\setminus D} = 0) = 1$ for all $y \in \partial D$.

• The stopped process at the boundary of $D$: $\overline{Y}_t := Y_{t \wedge T}$, where $T$ is the entry time of $\partial D$, $T := \inf\{t \geq 0 : Y_t \in \partial D\}$.

• Assume that $P^x(T < \zeta) = 1$ for all $x \in \overline{D}$. Let $c \in \text{bp}\beta(D)$ and extend it to $F$ with zero on $F \setminus D$. 
\( E := \overline{D} \)

**The base process** \( X \) on \( E \): the stopped process \( \overline{Y}_t \).

- Let \((T_t)_{t \geq 0}\) be the transition function of the process having \( \mathcal{L} - c \) as infinitesimal generator, it is expressed using the Feynman-Kac formula:

\[
T_t f(x) = E^x \left[ e^{-\int_0^t c(X_s) \, ds} f(X_t) \right], \quad f \in \text{bp}\mathcal{B}.
\]

- Since \( c \) equals zero on \( \partial D \) we get for \( x \in E \) and \( f \in \text{bp}\mathcal{B}(F) \)

\[
\exists \lim_{t \to \infty} T_t f(x) = E^x \left[ e^{-\int_0^T c(Y_s) \, ds} f(Y_T) \right] =: P^c_T f(x).
\]
Assumptions

(i) If $\varphi \in bpC(\partial D)$ then

$$\lim_{D \ni x \to y} P_T^c \varphi(x) = \varphi(y) \text{ for all } y \in \partial D.$$ 

(ii) The resolvent $\mathcal{W} = (W_\alpha)_{\alpha > 0}$ of $Y$ is strong Feller, i.e., $W_\alpha f \in C(F)$ for all $f \in bp\mathcal{B}(F)$ and $\alpha > 0$.

(iii) Domination property: If $f \in bp\mathcal{B}(F)$ and $v \in \mathcal{E}(\mathcal{W}_\beta)$, $v \leq W_\beta f$, then $v$ is continuous.
Proposition

Let $B : \text{bp}\beta(S) \longrightarrow \text{bp}\beta$ be a sub-Markovian kernel, $(\hat{H}_t)_{t \geq 0}$ the branching semigroup on $S$ constructed above, and $\mathcal{L}$ the infinitesimal generator of $X$. If $\varphi \in \text{bp}C(\partial D), \varphi \leq 1$, is such that

$$\begin{align*}
\exists \lim_{t \to \infty} H_t\varphi(x) =: u(x), \quad x \in E,
\end{align*}$$

then $u$ is a solution of the nonlinear Dirichlet problem

$$
\begin{cases}
(\mathcal{L} - c)u + cB\hat{u} = 0 \text{ on } D \\
\lim_{D \ni x \to y} u(x) = \varphi(y) \text{ for all } y \in \partial D.
\end{cases}
$$

Remark. Under certain additional assumptions, the existence of the limit (1) is proved in [M. Nagasawa 76], in the case when $B$ is given by $(\ast)$.