MODELLING ANTICIPATIONS ON FINANCIAL MARKETS

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Abstract. The aim of the present course is to give a review of the modern mathematical tools which can be used on a financial market by a "small" investor who possesses some informations on the price process.

Contents

Financial markets obviously have asymmetry of information. That is, there are different types of traders whose behavior is induced by different types of information that they possess.

Let us consider a "small" investor who trades in an arbitrage free financial market so as to maximize the expected utility of his wealth at a given time horizon. We assume that he is in the following position : He possesses extra information about some functional Y of the future prices of a stock (e.g. value of the price at a given date, hitting times of given values, ...). Our basic question is then: What is the value of this information ?

We can imagine two modelling approaches:

(1) A strong approach: The investor knows the functional ω by ω. This modelling of the additional information was initiated in [41] using initial enlargement of filtration, a theory developed in [27], [28], and [29].

(2) A weak approach: The investor knows the law of the functional Y under the effective probability of the market assumed to be unknown. This notion of weak information is defined in [5], [6], and further studied in [8].

We present and compare these two approaches.

1. Mathematical Framework

Let $T > 0$ be a constant finite time horizon. In what follows, we will work on a continuous arbitrage free financial market. Namely, we work on a filtered probability space $\left(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P}\right)$ which satisfies the usual conditions (i.e. $\mathcal{F}$ is complete and right-continuous and $\mathcal{F}$ is assumed to be trivial). We assume that the price process of a given contingent claim is a continuous $d-$dimensional and $\mathcal{F}$-adapted square integrable local martingale $(S_t)_{0 \le t \le T}$. In addition, we shall assume that the quadratic covariation matrix of the $d-$dimensional process S which is denoted by $\langle S \rangle$:

$$\langle S \rangle_t = \left(\langle S^i, S^j \rangle_t\right)_{1 \le i,j \le d}$$

is almost surely valued in the space of positive matrix, which means that S is non-degenerate. For a matrix M, M^* will denote the transpose of M and for a vector $v \in \mathbb{R}^d$, $\operatorname{diag}(v)$ denotes the $d \times d$ matrix

$$\begin{pmatrix} v_1 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & v_d \end{pmatrix}.$$

Of course, since S is a local martingale under $\mathbb{P}$, the market just defined has no arbitrage (precisely, there are no arbitrage opportunities with tame portfolios, see Corollary 2 in [36]; see also [15] for a general view on the absence of arbitrage property). Moreover, to take a null drift under $\mathbb{P}$, means that we consider discounted prices (or equivalently, prices expressed in the bond numéraire). The key point is that we start from the *observable* dynamics of S (a model of S under $\mathbb{P}$ can be calibrated to market data) and not from the "true" but unobservable dynamics of S under a so-called "historical" measure. Our modelling of anticipations on S will provide both in the strong and the weak approach an additional drift which will be calibrated over the information that the insider possesses. The quadratic variation process, or volatility, will be not be affected by anticipations on S; indeed, from a mathematical point of view this quadratic variation is invariant both by enlargements of filtration and equivalent changes of probability. Moreover, it is important to note that we could actually start from any semimartingale model for the price process.

Definition 1. *The space $\mathcal{M}(S)$ of martingale measures is the set of probabilities $\widetilde{\mathbb{P}} \sim \mathbb{P}$ such that $(S_t)_{0\leq t\leq T}$ is an $\mathcal{F}$-adapted local martingale under $\widetilde{\mathbb{P}}$.*

Let us now precise what we mean by arbitrage on the financial market

$$\left(\Omega, (\mathcal{H}_t)_{0\leq t<T}, (S_t)_{0\leq t<T}, \mathbb{Q}\right)$$

where $\mathcal{H}$ is a filtration (right-continuous and $\mathbb{P}-$complete) which contains the natural filtration of S, and $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$.

Definition 2. *We say that there is an arbitrage on the financial market*

$$\left(\Omega, (\mathcal{H}_t)_{0\leq t<T}, (S_t)_{0\leq t<T}, \mathbb{Q}\right),$$

if there exists a probability measure $\widetilde{\mathbb{Q}}$ equivalent to $\mathbb{Q}$ such that $(S_t)_{0\leq t<T}$ is an $\mathcal{H}$-adapted local martingale under $\widetilde{\mathbb{Q}}$.

Definition 3. *The space $\mathcal{A}_{\mathcal{F}}(S)$ of admissible strategies is the space of $\mathbb{R}^d-$valued and $\mathcal{F}-$predictable processes Θ integrable with respect to the price process S, such that*

$$\left(\int_0^t \Theta_u \cdot dS_u\right)_{0\leq t\leq T}$$

is a $\left(\widetilde{\mathbb{P}}, \mathcal{F}\right)$ martingale for all $\widetilde{\mathbb{P}}\in\mathcal{M}(S)$.

Remark 1. *Θ_t^i represents the number of shares of the risky asset S^i held by an investor at time t and the wealth process associated with the strategy $\Theta \in \mathcal{A}(S)$ with initial capital x is given by*

$$V_t = x + \int_0^t \Theta_u \cdot dS_u.$$

In particular, our strategies are self-financing.

Remark 2. *This set of admissible strategies is restrictive. We used it because of the Definition 6 of admissible strategies in an enlarged filtration. Nevertheless, we believe that is possible to extend most of the presented results in a slighty much general setting.*

We shall also very often assume that the financial market

$$\left(\Omega, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}, (S_t)_{0\leq t\leq T}\right)$$

is complete in the sense that the martingale $(S_t)_{0\leq t\leq T}$ enjoys the following predictable representation property (in abbreviate PRP) : For each $\mathcal{F}$-adapted local martingale $(M_t)_{0\leq t\leq T}$ there exists a predictable process Θ locally in L^2 such that

$$M_t = M_0 + \int_0^t \Theta_u \cdot dS_u, \quad t \leq T.$$

Remark 3. *Under the previous assumption, we have*

$$\mathcal{M}(S) = \{\mathbb{P}\}.$$

Indeed, if $\widetilde{\mathbb{P}} = D\ \mathbb{P} \in \mathcal{M}(S)$, then since $(S_t)_{0\leq t\leq T}$ is a local martingale under $\widetilde{\mathbb{P}}$, thanks to Girsanov's theorem we get that for all $1 \leq i \leq d$

$$d\langle S^i, D\rangle_t = 0.$$

But, by assumption, there exists Θ, locally in L^2, such that

$$D = 1 + \int_0^T \Theta_u \cdot dS_u$$

which implies that for all $1 \leq i \leq d$

$$\Theta_u^i = 0, \ 0 \leq u \leq T,$$

and hence

$$D = 1.$$

In what follows, we use the following notion of utility functions (see [32]):

Definition 4. *A utility function is a strictly increasing, strictly concave and twice continuously differentiable function*

$$U : (0, +\infty) \rightarrow \mathbb{R}$$

which satisfies

$$\lim_{x \rightarrow +\infty} U'(x) = 0, \quad \lim_{x \rightarrow 0^+} U'(x) = +\infty.$$

We use the convention that $U(x) = -\infty$ for $x \leq 0$. We shall denote by I the inverse of U', and by $\tilde{U}$ the convex conjugate of U:

$$\tilde{U}(y) = \max_{x>0} (U(x) - xy)$$

(that is, the Fenchel-Legendre transform of $-U(-x)$).

For a sake of simplicity, we limit ourselves to smooth utility functions, although it is also possible to obtain some results in the non-smooth case (see e.g. [10]). In our examples, we study the cases $U(x) = \ln(x)$ and $U(x) = x^{\alpha}$. The case $U(x) = e^{\alpha x}$ is also interesting; it does not fit into the previous definition, though our results remain true in this case.

Let us now introduce the object on which anticipations will be made.

Let $\mathcal{P}$ be a Polish space (for example $\mathcal{P} = \mathbb{R}^n$, $\mathcal{P} = C(\mathbb{R}_+, \mathbb{R}^n)$, etc...) endowed with its Borel $\sigma-$algebra $\mathcal{B}(\mathcal{P})$ and let $Y : \Omega \rightarrow \mathcal{P}$ be an $\mathcal{F}_T-$measurable random variable (it will be a functional of the trajectories of the price process). We denote by $\mathbb{P}_Y$ the law of Y and assume that Y admits a regular disintegration with respect to the filtration $\mathcal{F}$, precisely we assume that:

Assumption 1. *There exists a jointly measurable continuous in t and $\mathcal{F}-$adapted process*

$$\eta_t^y, \quad 0 \leq t < T, \quad y \in \mathcal{P},$$

satisfying for $dt \otimes \mathbb{P}_Y$ almost every $0 \leq t < T$ and $y \in \mathcal{P}$,

$$\mathbb{P}(Y \in dy \mid \mathcal{F}_t) = \eta_t^y \; \mathbb{P}_Y(dy). \tag{1.1}$$

This is a classical assumption of the theory of the initial enlargement of the filtration $\mathcal{F}$ by Y (see [27]) . This assumption is not very restrictive, and will be satisfied for *nice* functionals as it will be seen. The existence of a conditional density $\frac{\mathbb{P}(Y \in dy | \mathcal{F}_t)}{\mathbb{P}_Y(dy)}$ is the main point, the existence of a regular version follows from general results on stochastic processes (see [45]).

Remark 4. *We can note here that, if we denote by $\mathbb{P}^y$ the disintegrated probability measure defined by $\mathbb{P}^y = \mathbb{P}(\cdot \mid Y = y)$, then the above assumption implies that for $t < T$,*

$$\mathbb{P}^y_{/\mathcal{F}_t} = \eta^y_t \mathbb{P}_{/\mathcal{F}_t}.$$

In particular, for $\mathbb{P}_Y - a.e.$ $y \in \mathcal{P}$, the process $(\eta^y_t)_{0 \le t < T}$ is a martingale in the filtration $\mathcal{F}$ (not uniformly integrable, as it is nicely illustrated in [23]*).*

Finally, we shall denote by $\mathcal{G}$ the filtration $\mathcal{F}$ initially enlarged with Y, i.e. $\mathcal{G}_t$ is the $\mathbb{P}-$completion of $\bigcap_{\varepsilon>0} (\mathcal{F}_{t+\varepsilon} \vee \sigma(Y))$, $t < T$.

2. Strong Information Modelling

The theory of initial enlargement of filtration has been developed by the French school of probability during the eighties (see [27], [28], [29], [47] and [46]). This theory has many deep applications, most of which have been worked out by T. Jeulin and M. Yor. In the past few years we have seen new interest in this theory because of its applications in mathematical finance in the topic of the asymmetry of information. Papers where applications of the enlargement of filtration technique is applied to portfolio optimization of an insider include [2], [3], [18], [20], [24], [25], and [41]. In this chapter, we review the theory of initial enlargement of filtration and its applications in finance. We shall **always** assume that the financial market

$$\left(\Omega, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P}, (S_t)_{0 \le t \le T}\right)$$

is complete (i.e. that S enjoys the PRP, see section 1).

2.1. Some Results on Initial Enlargement of Filtration.

2.1.1. *Martingales of the Enlarged Filtration.* Let us start this subsection with some simple remarks about martingales in the initially enlarged filtration $\mathcal{G}$.

Proposition 1.

- *Let $(M_t)_{0 \le t < T}$ be an $\mathcal{F}-$adapted process which is a (local) martingale in the filtration $\mathcal{G}$, then it is also a (local) martingale in the filtration $\mathcal{F}$.*
- *Let $(M_t)_{0 \le t < T}$ be an integrable process adapted to the filtration $\mathcal{G}$, then the two following statements are equivalent:*

(1) *For $\mathbb{P}_Y -$ a.e. $y \in \mathcal{P}$, the process $\left(\mathbb{E}^y(M_t \mid \mathcal{F}_t)\right)_{0 \leq t < T}$ is a $\mathbb{P}^y-$(local) martingale in the filtration $\mathcal{F}$, $\mathbb{P}^y$ being the disintegrated probability measure defined by $\mathbb{P}^y = \mathbb{P}(\cdot \mid Y = y)$ and $\mathbb{E}^y$ the expectation under this measure.*

(2) *The process $(M_t)_{0 \leq t < T}$ is a (local) martingale in the filtration $\mathcal{G}$.*

Proof. We make the proof of this proposition with true martingales because the case of local martingales is easily deduced by localization. The first point is trivial, indeed for $s < t < T$

$$\mathbb{E}\left(M_t \mid \mathcal{G}_s\right) = M_s$$

which implies, by conditioning with respect to $\mathcal{F}_s$,

$$\mathbb{E}\left(M_t \mid \mathcal{F}_s\right) = M_s.$$

Let us now prove the second point. Since for $s < t < T$, $A \in \mathcal{F}_s$ and $\Lambda \in \mathcal{B}(\mathcal{P})$ we have

$$\mathbb{E}\left(\left(M_t - M_s\right) 1_{A \cap (Y \in \Lambda)}\right) = \int_\Lambda \mathbb{E}^y\left(\left(M_t - M_s\right) 1_A\right) \mathbb{P}_Y(dy)$$

it is easily seen by a monotone class theorem that our equivalence takes place. □

Of course, (local) martingales in the filtration $\mathcal{F}$ do not remain (local) martingales in the enlarged filtration, but as shown in the following section, they remain semimartingales.

2.1.2. *Jacod's theorem.* Jacod's celebrated theorem says that a semimartingale which is adapted to the filtration $\mathcal{F}$ remains a semimartingale in the enlarged filtration $\mathcal{G}$. Before we state it, we start with a representation lemma which allows to give explicitly the semimartingale decomposition of S in the enlarged filtration. In what follows, $\mathcal{P}(\mathcal{F})$ denotes the predictable σ-field associated with the filtration $\mathcal{F}$.

Lemma 1. *(See [27]) There exists a $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathcal{P})$ measurable process*

$$\begin{array}{ccccccc} [0,T[& \times & \Omega & \times & \mathcal{P} & \rightarrow & \mathbb{R}^d \\ (\,t & , & \omega & , & y\,) & \rightarrow & \alpha_t^y(\omega) \end{array}$$

such that:

(1) *For $\mathbb{P}_Y -$ a.e. $y \in \mathcal{P}$ and for $0 \leq t < T$, $1 \leq i \leq d$,*

$$\mathbb{P}\left(\int_0^t (\alpha_u^y)^* \, d\langle S\rangle_u \alpha_u^y < +\infty\right) = 1$$

(2) *For $\mathbb{P}_Y -$ a.e. $y \in \mathcal{P}$ and for $0 \leq t < T$, $1 \leq i \leq d$,*

$$\langle \eta^y, S^i \rangle_t = \int_0^t \eta_u^y \left(\sum_{i=1}^d \alpha_u^{y,j} d\langle S^i, S^j\rangle_u\right).$$

Remark 5. *We can choose α such that for $\mathbb{P}_Y - a.e.$ $y \in \mathcal{P}$ and for $0 \le t < T$,*

$$\eta_t^y = \exp\left(\int_0^t \alpha_u^y \cdot dS_u - \frac{1}{2}\int_0^t (\alpha_u^y)^* \, d\langle S\rangle_u \alpha_u^y\right) \text{ on } \{\eta_t^y > 0\}.$$

We shall need (actually, only in the next section on the modelling of a weak anticipation; without this assumption Jacod's theorem remains true) the following additional assumption which ensures the existence of a version of α which allows to use filtering theory:

Assumption 2. *For almost every $t \in [0,T)$, we have $\mathbb{P}$-almost surely*

$$\int_0^t d\langle S^i, S^j\rangle_u \mathbb{E}\left(\left\|\alpha_u^Y\right\| \mid \mathcal{F}_u\right)^2 < +\infty, \quad 1 \le i,j \le d.$$

Theorem 1. *(See [27]) Under the probability $\mathbb{P}$ the price process $(S_t)_{0 \le t < T}$ is a semimartingale in the filtration $\mathcal{G}$, and its decomposition is given by*

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \alpha_u^Y + M_t, \qquad 0 \le t < T \tag{2.1}$$

where $(M_t)_{0 \le t < T}$ is a local martingale in the filtration $\mathcal{G}$ such that

$$\langle S\rangle = \langle M\rangle.$$

Proof. From Proposition 1, it is enough to show that for *a.e.* $y \in \mathcal{P}$, the process

$$M_t^y := -\int_0^t d\langle S\rangle_u \alpha_u^y + S_t$$

is a $\mathbb{P}^y-$local martingale in the filtration $\mathcal{F}$. But this is a direct consequence of Girsanov's theorem, because $(\eta_t^y)_{0 \le t < T}$ is the density process of $\mathbb{P}^y$ with respect to $\mathbb{P}$ (see Remark 4). □

Hence the class of semimartingales is preserved by an initial enlargement of filtration. Related to this is Stricker's theorem (see [44]): If a process is a semimartingale in an enlarged filtration, then it is a semimartingale in its own filtration.

We conclude this paragraph with a converse of Jacod's theorem. As it will be seen later, this theorem makes the link between the strong and the weak approach and also shows that the natural filtration of M (as defined in the previous Theorem) is strictly included in $\mathcal{G}$.

Theorem 2. *(See [5]) Let $\mathbb{Q}$ be a probability measure on Ω which is equivalent to $\mathbb{P}$ with a bounded density. If the process $(M_t)_{0 \le t < T}$ defined by*

$$M_t := S_t - \int_0^t d\langle S\rangle_u \alpha_u^Y, \qquad 0 \le t < T$$

is a local martingale under $\mathbb{Q}$ in the filtration $\mathcal{G}$, then there exists a probability measure ν on $\mathcal{P}$ such that

$$\mathbb{Q} = \int_{\mathcal{P}} \mathbb{P}\left(\cdot \mid Y = y\right) \nu\left(dy\right).$$

Proof. For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, let us denote by $\mathbb{Q}^y$ the conditional probability $\mathbb{Q}\left(\cdot \mid Y = y\right).$ From our assumption and Proposition 1, the process M is, under $\mathbb{Q}^y$, a local martingale. Now, because $\mathbb{Q}$ is assumed to be equivalent to $\mathbb{P}$, it is easily seen that for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, $\mathbb{Q}^y$ is locally absolutely continuous on $\mathcal{F}$ with respect to $\mathbb{P}$. Hence, by Girsanov's theorem, for $\mathbb{P}_Y - a.e.$ $y \in \mathcal{P}$,

$$d\mathbb{Q}^y_{/\mathcal{F}_t} = \eta^y_t d\mathbb{P}_{/\mathcal{F}_t}\ ,\ t < T.$$

Since we also have, for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$,

$$d\mathbb{P}^y_{/\mathcal{F}_t} = \eta^y_t d\mathbb{P}_{/\mathcal{F}_t}\ ,\ t < T,$$

as explained in Remark 4, where $\mathbb{P}^y$ is the conditional probability $\mathbb{P}\left(\cdot \mid Y = y\right)$, we immediately deduce

$$\mathbb{Q}^y = \mathbb{P}^y$$

for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, and hence

$$\mathbb{Q} = \int_{\mathcal{P}} \mathbb{P}\left(\cdot \mid Y = y\right) \nu\left(dy\right)$$

where ν is the law of Y under $\mathbb{Q}$. □

2.1.3. *Martingale Preserving Measure and PRP in the Enlarged Filtration.* In all this paragraph, we shall assume that for $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, the process $(\eta^y_t)_{0 \le t < T}$ is strictly positive $\mathbb{P}$-a.s.

Lemma 2. *(See* [3]*) The process*

$$Z^Y_t := \frac{1}{\eta^Y_t},\ t < T$$

is a $\mathbb{P}$-martingale (not necessarily uniformly integrable) in the enlarged filtration $\mathcal{G}$. Moreover, it satisfies

$$\mathbb{E}\left(Z^Y_t \mid \mathcal{F}_t\right) = 1,\ t < T. \tag{2.2}$$

Proof. For $\mathbb{P}_Y - a.e.\ y \in \mathcal{P}$, the process $(Z^y_t)_{0 \le t < T}$ is a $\mathbb{P}^y-$martingale in the filtration $\mathcal{F}$ because it is the density process of $\mathbb{P}^y$ with respect to $\mathbb{P}$. We conclude with Proposition 1. Now, for the second point, we claim that

$$\mathbb{E}\left(Z^Y_t \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} Z^y_t \eta^y_t \mathbb{P}_Y\left(dy\right) = 1,$$

which completes the proof. □

Definition 5. *The probability measure $\widetilde{\mathbb{P}}_t$, $t<T$, defined on $\mathcal{G}_t$ by*

$$\widetilde{\mathbb{P}}_t = Z_t^Y \ \mathbb{P}_{/\mathcal{G}_t}$$

is called the $[0,t]-$martingale preserving measure associated with $\mathbb{P}$.

Remark 6. *As a consequence of (2.2), we can note that for $t<T$ the law of an $\mathcal{F}-$adapted process $(X_s)_{0\le s\le t}$ is the same under $\mathbb{P}_{/\mathcal{F}_t}$ as under $\widetilde{\mathbb{P}}_t$.*

Remark 7. *We have the following representation result*

$$Z_t^Y = \exp\left(-\int_0^t \alpha_u^Y \cdot dM_u - \frac{1}{2}\int_0^t \left(\alpha_u^Y\right)^* d\langle M\rangle_u \alpha_u^Y\right),\ t<T.$$

The terminology of martingale preserving measure stems from the following theorem.

Theorem 3. *(See [3]) For $t<T$, any $\mathbb{P}-$(local) martingale adapted to $(\mathcal{F}_s)_{0\le s\le t}$ is also a $\widetilde{\mathbb{P}}_t-$(local) martingale in the enlarged filtration $(\mathcal{G}_s)_{0\le s\le t}$and thus a $\widetilde{\mathbb{P}}_t-$(local) martingale in the filtration $(\mathcal{F}_s)_{0\le s\le t}$.*

which is a consequence of the following easy lemma.

Lemma 3. *(See [3]) For $t<T$, under the probability $\widetilde{\mathbb{P}}_t$, the $\sigma-$algebras $\mathcal{F}_t$ and $\sigma(Y)$ are independent.*

We conclude this subsection on the general theory of initial enlargement of a filtration with a particular case of the PRP of the martingale preserving measure in the enlarged filtration obtained by Amendinger [1]. This representation result is an easy consequence of the stability of the PRP by Girsanov's transforms (see Proposition 17.1. in [47]).

Theorem 4. *Let $t<T$. For any $\widetilde{\mathbb{P}}_t-$(local) martingale $(M_s)_{0\le s\le t}$ adapted to $(\mathcal{G}_s)_{0\le s\le t}$ and which satisfies $M_0=0$, there exists a $\mathcal{G}-$predictable process Θ which is integrable with respect to S and which satisfies*

$$M_s = \int_0^s \Theta_u dS_u,\ s\le t.$$

2.2. **Examples of Initial Enlargement of Filtration.** We present now some illustrative examples of enlargement of filtration. For further examples, we refer the interested reader to the "Récapitulatif" in [29], pp. 305-313 and Chapter 12 of [47].

2.2.1. *Stochastic Analysis Warm Up.* Here, we use stochastic analysis to obtain explicit computations for the enlargement formula of the natural filtration of the coordinate process. First, we recall some basic definitions of stochastic analysis (for further details we refer to the book [40]). Consider the $d-$dimensional Wiener space of continuous paths

$$\mathbb{W}^d = \left(\mathcal{C}_T^d, (\mathcal{F}_t)_{0\leq t\leq T}, (X_t)_{0\leq t\leq T}, \mathbb{P}\right)$$

where:

(1) $\mathcal{C}_T^d$ is the space of continuous functions $f:[0,T]\to\mathbb{R}^d$, such that $f(0)=0$
(2) $(X_t)_{0\leq t\leq T}$ is the coordinate process defined by $X_t(f)=f(t)$
(3) $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of $(X_t)_{0\leq t\leq T}$
(4) $\mathbb{P}$ is the Wiener measure.

We assume furthermore that

$$\mathcal{P}=\mathbb{R}^N$$

for some integer $N\geq 1$ and that Y belongs to $\left(\mathbb{D}^{1,2}\right)^N$. We recall (see [40] pp. 26) that the Hilbert space $\mathbb{D}^{1,2}$ is the closure of the class of smooth cylindric random variables with respect to the norm

$$\|F\|_{1,2}=\left(\mathbb{E}\left(F^2\right)+\mathbb{E}\left(\|\mathbf{D}F\|_{L^2}^2\right)\right)^{\frac{1}{2}}$$

where $\mathbf{D}$ is the gradient defined for a functional $F=\Phi\left(X\left(f_1\right),...,X\left(f_d\right)\right)$ by

$$\mathbf{D}F=\left(Jac\phi\right)\left(f_1,...,f_d\right)$$

where $\Phi:\mathbb{R}^d\to\mathbb{R}^d$ is a smooth function whose Jacobian matrix is denoted $Jac\phi$, and $(f_i)_{i=1,\dots,d}\in\left(L^2\right)^d$. We give now in this setting, by means of the Clark-Ocone formula, some expressions for α and η in the case where $S_t=X_t$, $t\geq 0$.

Theorem 5. *(see* [5]*) Assume that for almost every $t<T$,*

$$\int_{\mathbb{R}^N}\left|\mathbb{E}\left(e^{i\lambda\cdot Y}\mid\mathcal{F}_t\right)\right|d\lambda<+\infty \tag{2.3}$$

then Y has a density p with respect to the Lebesgue measure which is given by

$$p(y)=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-iy\cdot\lambda}\,\mathbb{E}\left(e^{i\lambda\cdot Y}\right)d\lambda,$$

and for $\mathbb{P}_Y-a.e.$ $y\in\mathbb{R}^N$ and for almost every $t<T$,

$$p(y)\;\eta_t^y=\frac{1}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-iy\cdot\lambda}\,\mathbb{E}\left(e^{i\lambda\cdot Y}\mid\mathcal{F}_t\right)d\lambda. \tag{2.4}$$

Moreover, if for $\mathbb{P}_Y - a.e$ $y \in \mathbb{R}^N$ and for almost every $t < T$,

$$\int_{\mathbb{R}^N} \left\| \mathbf{D}_t \mathbb{E} \left(e^{i\lambda \cdot Y} \mid \mathcal{F}_t \right) \right\| d\lambda < +\infty, \ p(y) \neq 0,$$

then for $\mathbb{P}_Y - a.e$ $y \in \mathbb{R}^N$ and for almost every $t < T$, $\eta_t^y \in Dom\,(\mathbf{D})$ and

$$\mathbf{D}_t \eta_t^y = \eta_t^y \alpha_t^y. \tag{2.5}$$

Proof. Let m be a function on $\mathbb{R}^N$ such that

$$\int_{\mathbb{R}^N} |m(y)|\, dy < +\infty$$

Let now $\widetilde{m}$ be the Fourier transform of m defined on $\mathbb{R}^N$ by

$$\widetilde{m}\,(y) = \int_{\mathbb{R}^N} e^{iy\cdot\lambda} m\,(d\lambda)\,.$$

We have for $t < T$,

$$\int_{\mathbb{R}^N} \eta_t^y \ \widetilde{m}\,(y)\, \mathbb{P}_Y\,(dy) = \mathbb{E}\left(\widetilde{m}\,(Y) \mid \mathcal{F}_t\right).$$

But,

$$\mathbb{E}\left(\widetilde{m}\,(Y) \mid \mathcal{F}_t\right) = \int_{\mathbb{R}^N} \mathbb{E}\left(e^{iY\cdot\lambda} \mid \mathcal{F}_t\right) m\,(\lambda)\, d\lambda$$

hence,

$$\int_{\mathbb{R}^N} \eta_t^y \ \widetilde{m}\,(y)\, \mathbb{P}_Y\,(dy) = \int_{\mathbb{R}^N} \mathbb{E}\left(e^{iY\cdot\lambda} \mid \mathcal{F}_t\right) m\,(\lambda)\, d\lambda.$$

Since the previous equality takes place for all m, this implies, thanks to the inversion formula for the Fourier transform:

$$m\,(\lambda) = \frac{1}{(2\pi)^N} \int_{\mathbb{R}^N} e^{-iy\cdot\lambda} \widetilde{m}\,(y)\, dy$$

that for $\mathbb{P}_Y - a.e$ $y \in \mathbb{R}^N$ and for all $t < T$,

$$\eta_t^y \ \mathbb{P}_Y(dy) = \frac{1}{(2\pi)^N} \left(\int_{\mathbb{R}^N} e^{-iy\cdot\lambda} \ \mathbb{E}\left(e^{i\lambda\cdot Y} \mid \mathcal{F}_t\right) d\lambda \right) dy.$$

This provides easily the first part of our theorem.

Assume now that for $\mathbb{P}_Y - a.e.$, $y \in \mathbb{R}^N$ and for $a.e.$ $t < T$,

$$\int_{\mathbb{R}^N} \left\| \mathbf{D}_t \mathbb{E} \left(e^{i\lambda \cdot Y} \mid \mathcal{F}_t \right) \right\| d\lambda < +\infty.$$

Because for for $\mathbb{P}_Y - a.e.$, $y \in \mathbb{R}^N$ and for $a.e.$ $t < T$, $\lambda \in \mathbb{R}$,

$$\mathbb{E}\left(e^{i\lambda\cdot Y} \mid \mathcal{F}_t\right) \in \mathbb{D}^{1,2}$$

it is easily seen that $\eta_t^y \in Dom\left(\mathbf{D}\right)$. Moreover, let us consider a probability measure μ on $\mathbb{R}^N$ such that μ is equivalent to $\mathbb{P}_Y$ and $\xi := \frac{d\mu}{d\mathbb{P}_Y}$ admits a bounded continuously differentiable version. From the Clark-Ocone formula, we have:

$$d\left[\int_{\mathbb{R}^N} \eta_t^y \mu\left(dy\right)\right] = \mathbb{E}\left(\mathbf{D}_t \xi\left(Y\right) \mid \mathcal{F}_t\right) \cdot dX_t$$

which implies,

$$\int_{\mathbb{R}^N} \alpha_t^y \eta_t^y \mu\left(dy\right) = \mathbf{D}_t \mathbb{E}\left(\xi\left(Y\right) \mid \mathcal{F}_t\right) = \mathbf{D}_t \int_{\mathbb{R}^N} \eta_t^y \mu\left(dy\right)$$

for a.e. $t \in [0, T]$ and the conclusion follows easily because μ was arbitrary. □

Remark 8. *Of course, formula* (2.4) *remains true under the assumption* (2.3) *even if* $Y \notin \left(\mathbb{D}^{1,2}\right)^N$.

Remark 9. *The formula 2.5 can be found in* [26] *(Proposition A.1.) in an equivalent form. Indeed, in this paper the authors have developed a Malliavin Calculus for measure valued random variables and gave a sense to the following formula*

$$\alpha_t^y = \frac{\mathbb{E}(\mathbf{D}_t \delta_Y \mid \mathcal{F}_t)(dy)}{\mathbb{P}(Y \in dy \mid \mathcal{F}_t)}.$$

2.2.2. *Initial Enlargement with the Terminal Value of a Diffusion.* In this subsection, we consider the case $\mathcal{P} = \mathbb{R}^d$ and $Y = S_T$. We assume furthermore that the dynamics of $(S_t)_{0 \leq t \leq T}$ under $\mathbb{P}$ are given by

$$S_t = s_0 + \int_0^t \mathrm{diag}\left(S_u\right) \sigma\left(S_u\right) dB_u, \qquad 0 \leq t \leq T \tag{2.6}$$

where $(B_t)_{0 \leq t \leq T}$ is a d-dimensional standard Brownian motion whose filtration is $\mathcal{F}$, $s_0 \in \left(\mathbb{R}_+^*\right)^d$ and σ a positive definite symmetric bounded C^∞ function with bounded partial derivatives function satisfying

$$\inf_{x \in \mathbb{R}^d} \ \left\|\sigma\sigma^*\left(x\right)\right\| \geq a > 0.$$

The last assumption implies (Hörmander's theorem, see [40] Chap. 2) the differentiability of the transition function p_t, $0 < t \leq T$, defined by

$$p_t\left(x, y\right) dy = \mathbb{P}\left(S_t \in dy \mid S_0 = x\right).$$

Moreover, it is easily seen that we obviously are in the assumptions of the theory of the initial enlargement of $\mathcal{F}$ with S_T. Namely, for $0 \leq t < T$,

$$\mathbb{P}\left(S_T \in dy \mid \mathcal{F}_t\right) = p_{T-t}\left(S_t, y\right) dy$$

which implies that for $\mathbb{P}_{S_T} - a.e.\ y \in \mathbb{R}^d$ and $t < T$

$$\eta_t^y = \frac{p_{T-t}(S_t, y)}{p_T(s_0, y)}. \tag{2.7}$$

In what follows, for a smooth function $f(x, y)$ defined on $\mathbb{R}^d \times \mathbb{R}^d$, ∇f denotes the gradient computed with respect to the first variable.

Theorem 6. *In the filtration $\mathcal{G}$, the process $(S_t)_{0 \le t < T}$ admits the following semimartingale decomposition*

$$S_t = s_0 + \int_0^t d\langle S \rangle_u \frac{\nabla p_{T-t}}{p_{T-t}}(S_u, S_T) + \int_0^t diag(S_u)\, \sigma(S_u)\, d\beta_u$$

where $(\beta_t)_{0 \le t \le T}$ is a d-dimensional standard Brownian motion adapted to the filtration $\mathcal{G}$ (and hence independent of S_T).

Proof. In this case, from the Markov property of S (see (2.7), or [21] for further details), we have for $\mathbb{P}_{S_T} - a.e.\ y \in \mathbb{R}^d$

$$\eta_t^y = \frac{p_{T-t}(S_t, y)}{p_T(s_0, y)},\ t < T.$$

Hence, thanks to Itô's formula

$$\alpha_t^y = \frac{\nabla p_{T-t}}{p_{T-t}}(S_t, y),$$

which leads, according to Jacod's theorem, to the expected result. □

2.2.3. *Initial Enlargement with the First Hitting Time of a Level of the Brownian Motion.* In this paragraph, we give the formula for the enlargement with the first hitting time of $a > 0$ by a standard Brownian motion B. Let us denote by T_a this stopping time, i.e.

$$T_a = \inf_{t \ge 0}\{t,\ B_t = a\}.$$

In this case, the assumption of the existence of η is not satisfied. Nevertheless, as it is seen in the proof of Theorem 7, there exists a jointly measurable continuous in t and $\mathcal{F}_{\cdot \wedge T_a}-$adapted process

$$\eta_t^y, \quad 0 \le t < T_a, \quad y \in \mathbb{R}_+^*$$

satisfying for $dt \otimes \mathbb{P}_{T_a}$ almost every $0 \le t < T_a$ and $y \in \mathbb{R}_+^*$,

$$\mathbb{P}(T_a \in dy \mid \mathcal{F}_t,\ t < T_a) = \eta_t^y\ \mathbb{P}(T_a \in dy).$$

Moreover, a process α can be associated (up to time T_a) with η in the same way as in the Lemma 1. In [28], T. Jeulin found the following enlargement formula:

Theorem 7. *In the filtration $\mathcal{G}=\mathcal{F}_{\cdot\wedge T_a}\vee\sigma\left(T_a\right)$, the process B admits the following semimartingale decomposition*

$$B_t=\beta_t-\int_0^t\left(\frac{1}{a-B_s}-\frac{a-B_s}{T_a-s}\right)ds,\quad t<T_a \tag{2.8}$$

where β is a standard Brownian motion adapted to the filtration $\mathcal{G}$ (and hence independent of T_a).

Proof. It is easily shown, by using exponential martingales that for $\alpha\in\mathbb{R}$,

$$\mathbb{E}\left(e^{-\frac{\alpha^2}{2}T_a}\mid\mathcal{F}_t,\ t<T_a\right)=e^{\alpha(B_t-a)-\frac{\alpha^2}{2}t}.$$

It stems from this, by inverting the previous Laplace transform that for $y>0$,

$$\frac{\mathbb{P}\left(T_a\in dy\mid\mathcal{F}_t,\ t<T_a\right)}{\mathbb{P}\left(T_a\in dy\right)}=\frac{a-B_t}{a}\left(\frac{y}{y-t}\right)^{3/2}e^{\frac{a^2}{2y}-\frac{(a-B_t)^2}{2(y-t)}}$$

which gives the expected result after straightforward computations. □

2.2.4. *Initial Enlargement with the Perpetuity.* We consider now the case $\mathcal{P}=\mathbb{R}$ and assume that the dynamics of S under $\mathbb{P}$ are one-dimensional and given by

$$S_t=\exp\left(B_t-\frac{1}{2}t\right),\ t\geq 0$$

where $(B_t)_{0\leq t\leq T}$ is a one-dimensional standard Brownian motion whose filtration is $\mathcal{F}$. We recall that the functional

$$Y=\int_0^{+\infty}S_t^2dt=\int_0^{+\infty}e^{2B_t-t}dt$$

is well defined and distributed, up to a multiplicative constant, as the inverse of a gamma law, precisely (see [19]):

$$\int_0^{+\infty}S_t^2dt\sim\frac{1}{2\gamma_{1/2}},$$

which means that

$$\mathbb{P}\left(\int_0^{+\infty}S_t^2dt\in dy\right)=\frac{1}{\sqrt{2}\Gamma\left(\frac{1}{2}\right)}\frac{e^{-\frac{1}{2y}}}{y^{3/2}}1_{y>0}dy.$$

We have then the following enlargement formula.

Theorem 8. *In the filtration $\mathcal{G}=\mathcal{F}\vee\sigma\left(\int_0^{+\infty}S_t^2dt\right)$, the process S admits the following semimartingale decomposition*

$$S_t=1+\int_0^tS_u\left(1-\frac{S_u^2}{\int_u^{+\infty}S_v^2dv}\right)du+\int_0^tS_ud\beta_u\ ,\ t\geq 0 \tag{2.9}$$

where $(\beta_t)_{t\geq 0}$ is a standard Brownian motion adapted to the enlarged filtration (and hence independent of $\int_0^{+\infty}S_t^2dt$).

Proof. From the Dubins-Schwarz theorem (which appears here as a special case of the Lamperti's relation, see [43] pp. 452), there exists a standard Brownian motion γ such that

$$S_t = 1 - \gamma_{\int_0^t S_u^2 du}.$$

Hence,

$$\int_0^{+\infty} S_t^2 dt = T_1$$

where

$$T_1 = \inf_{t \geq 0} \{t, \gamma_t = 1\}$$

and we conclude with the enlargement formula (2.8). □

Remark 10. *The decomposition* (2.9) *implies (see* [7]*)*

$$S_t = \frac{e^{\beta_t + \frac{1}{2}t}}{1 + \frac{\int_0^t e^{2\beta_s + \mu s} ds}{\int_0^{+\infty} S_t^2 dt}} \;, \; t \geq 0.$$

2.3. **Utility Maximization with Strong Information.**

2.3.1. *The Financial Market of the Informed Insider.* We now apply to finance the general results of initial enlargement of filtration presented in the previous sections. For this, we assume that for *a.e.* $y \in \mathcal{P}$, the process $(\eta_t^y)_{0 \leq t < T}$ is strictly positive $\mathbb{P}$-a.s. Hence, we can use the martingale preserving measure introduced earlier in Section 2.1.3. we study here the financial market

$$\left(\Omega, (\mathcal{G}_t)_{0 \leq t < T}, \mathbb{P}, (S_t)_{0 \leq t < T}\right) \tag{2.10}$$

and solve the portfolio optimization problem associated with this model. The first remark is the following (see Definition 2).

Proposition 2. *For $t < T$, there is no arbitrage in the time interval $[0, t]$. But there is an arbitrage in the time interval $[0, T]$.*

Proof. Indeed, from Theorem 3, the martingale preserving measure $\widetilde{\mathbb{P}}_t$ is a martingale measure for S. Now, let us assume that there exists a probability measure $\mathbb{Q}$ on $\mathcal{F}_T$ which is equivalent to $\mathbb{P}$ and such that S is a martingale under $\mathbb{Q}$. In this case, on the time-interval $[0, t]$, S is a local martingale. It implies from Theorem 4 that $\mathbb{Q}_{/\mathcal{G}_t} = \widetilde{\mathbb{P}}_t$, which can not happen because the $\mathbb{P}-$martingale $\left(\frac{1}{\eta_t^Y}\right)_{0 \leq t < T}$ is not uniformly integrable. □

2.3.2. *Value of the Strong Information.* Let now U be a utility function as defined in Section 1.

Definition 6. *The space $\mathcal{A}_{\mathcal{G}}(S)$ of admissible strategies is the space of $\mathbb{R}^d-$valued and $\mathcal{G}-$predictable processes $(\Theta_s)_{s<T}$ integrable with respect to the price process S, such that*

$$\left(\frac{1}{\eta_t^Y}\int_0^t \Theta_u \cdot dS_u\right)_{0\leq t<T}$$

is a $(\mathbb{P},\mathcal{G})$-martingale.

With this set of admissible strategies, we classicaly associate the following portfolio optimization problem.

Portfolio optimization problem on $[0,t]$: *For $t \in [0,T]$, the insider's portfolio optimization problem on $[0,t]$ is to find*

$$u(x,Y,t) := \sup_{\Theta\in\mathcal{A}_{\mathcal{G}}(S)} \mathbb{E}\left(U\left(x+\int_0^t \Theta_u dS_u\right)\right),$$

$x>0$ being the initial endowment of the insider.

We restrict ourselves to the time-interval $[0,t]$, because of the presence of the arbitrage discussed in the previous proposition and now solve the portfolio optimization problem on $[0,t]$.

Theorem 9. *(See [2]) Let $t\in[0,T)$, $x>0$ and let us assume that there exists an $\sigma(Y)-$measurable random variable $\Lambda_t(x):\Omega\to(0,+\infty)$ with*

$$\mathbb{E}\left(\frac{1}{\eta_t^Y} I\left(\frac{\Lambda_t(x)}{\eta_t^Y}\right) \mid Y\right) = x,$$

then

$$u(x,Y,t) = \mathbb{E}\left(\ (U\circ I)\left(\frac{\Lambda_t(x)}{\eta_t^Y}\right)\right).$$

Proof. Let us set

$$V_t = I\left(\frac{\Lambda_t(x)}{\eta_t^Y}\right).$$

First, we note that, according to Theorem 4, there exist $\Theta\in\mathcal{A}_{\mathcal{G}}(S)$ such that

$$V_t = x+\int_0^t \Theta_u\cdot dS_u.$$

Since U is concave, we have

$$U(b)\geq U(a)+U'(b)(b-a),\ \ a,b\in(0,+\infty).$$

Hence

$$U(V_t) \geq U\left(\widetilde{V}_t\right) + \frac{\Lambda_t(x)}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right)$$

where

$$\widetilde{V}_t = x + \int_0^t \widetilde{\Theta}_u \cdot dS_u,$$

with

$$\widetilde{\Theta} \in \mathcal{A}_{\mathcal{G}}(S).$$

Even if $\frac{\Lambda_t(x)}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right)$ is not integrable, we can take generalized conditional expectations to obtain

$$\mathbb{E}\left(\frac{\Lambda_t(x)}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right) \mid Y\right) = \Lambda_t(x)\,\mathbb{E}\left(\frac{1}{\eta_t^Y}\left(V_t - \widetilde{V}_t\right) \mid Y\right) = 0.$$

We conclude hence

$$\mathbb{E}\left(U(V_t) \mid Y\right) \geq \mathbb{E}\left(U\left(\widetilde{V}_t\right) \mid Y\right),$$

which gives the expected result. □

As an illustration of the previous theorem, we give the optimal expected utility in the case of the most commonly used utility functions.

Example 1. *Let* $\alpha \in (0,1)$ *and* $U(x) = \frac{x^\alpha}{\alpha}$ *, then*

$$I(y) = y^{\frac{1}{\alpha-1}},$$

$$\Lambda_t(x) = \frac{x^{\alpha-1}}{\mathbb{E}\left(\left(\eta_t^Y\right)^{\frac{\gamma}{1-\gamma}} \mid Y\right)},$$

and

$$u(x,Y,t) = \frac{x^\alpha}{\alpha}\mathbb{E}\left(\mathbb{E}\left[\left(\eta_t^Y\right)^{\frac{\alpha}{1-\alpha}} \mid Y\right]^{1-\alpha}\right).$$

Example 2. *Let* $U(x) = \ln x,$ *then*

$$I(y) = \frac{1}{y},$$

$$\Lambda_t(x) = \frac{1}{x},$$

and

$$u(x,Y,t) = \ln x + \mathbb{E}\left(\ln \eta_t^Y\right).$$

2.4. **Comments.** The fact that there always is an arbitrage in the financial market

$$\left(\Omega, (\mathcal{G}_t)_{0\leq t<T}, \mathbb{Q}, (S_t)_{0\leq t<T}\right)$$

is one of the principal problem of this approach. From the mathematical point of view, it is easily understood because the knowledge of a functional ω by ω is very restrictive. In Chapter 3, we develop the notion of weak information, which is much more flexible and which leaves more freedom on the model used by the informed insider.

Nevertheless, by a change of filtration, there is some possibilities to allow much freedom on the anticipation. For instance, we can enlarge the "public" filtration $\mathcal{F}$ by $Y+N$ where N is a noise independent of $\mathcal{F}$ and constant in the time. The computations and theorems associated with this kind of enlargement are studied in a very general setting in [2], and [1] for the PRP properties. Another possibility is to use a part of the theory of progressive enlargement of filtration (for further details on it we refer to [47] Section 12.2.). Roughly speaking, we would like that the "noise" N could evolve in the time. Typically, it is natural to study enlargement formulas associated with the filtration $\mathcal{F}_t \vee \sigma\left(Y + W_{T-s}, s\leq t\right)$ where W is a standard Brownian motion independent of $\mathcal{F}_T$. This idea is presented in [14], where the authors give, more generally, an enlargement formula associated with a filtration of the form $\mathcal{F}_t \vee \sigma\left(F(Y,W_s), s\leq t\right)$, W being this time any process independent of $\mathcal{F}_T$ and F a Borel function. In this setting, the authors proved that if the rate at which the additional noise in the insider's information vanishes is slow enough, then there is no arbitrage and the additional utility of the insider is finite. Let us see briefly more precisely what kind of "progressive" enlargement formulas can be obtained in full generality. Consider a filtration $\mathcal{W}$ independent of $\mathcal{F}_T$, and $\mathcal{H}$ a sub-filtration of $\mathcal{G}\vee\mathcal{W}$ which contains $\mathcal{F}$. We have then the following enlargement formula.

Theorem 10. *Assume that for almost every $t\in[0,T)$, we have $\mathbb{P}$-almost surely*

$$\int_0^t d\langle S^i, S^j\rangle_u \left\|\mathbb{E}\left(\alpha_u^Y \mid \mathcal{H}_u\right)\right\| < +\infty, \quad 1\leq i,j\leq d$$

then, under the probability $\mathbb{P}$ the price process $(S_t)_{0\leq t<T}$ is a semimartingale in the filtration $\mathcal{H}$, and its decomposition is given by

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \mathbb{E}\left(\alpha_u^Y \mid \mathcal{H}_u\right) + M_t, \qquad 0\leq t<T \tag{2.11}$$

where $(M_t)_{0\leq t<T}$ is a local martingale such that

$$\langle S\rangle = \langle M\rangle.$$

This theorem is easily understood by the "méthode des Laplaciens approchés" (see [16]). Namely, the bounded variation part A of a semimartingale N in the filtration $\mathcal{H}$ is

$$A_t = \lim_{h \rightarrow > 0} \int_0^t \frac{\mathbb{E}\left(N_{s+h} - N_s \mid \mathcal{H}_s\right)}{h} ds$$

as soon as the right hand exists in L^1, and we know that S is semimartingale in the filtration $\mathcal{H}$ because it is in the filtration $\mathcal{G} \vee \mathcal{W}$ from Jacod's theorem. From a financial point of view, we can note that the financial market

$$\left(\Omega, (\mathcal{H}_t)_{t<T}, (S_t)_{t<T}, \mathbb{P}\right)$$

is not complete, contrary to the market

$$\left(\Omega, (\mathcal{G}_t)_{t<T}, (S_t)_{t<T}, \mathbb{P}\right).$$

Indeed, it is clear that there are many martingale measures for S. For instance,

$$\mathbb{Q}_{/\mathcal{H}_t} = \mathbb{E}\left(\frac{D_t}{\eta_t^Y} \mid \mathcal{H}_t\right) \mathbb{P}_{/\mathcal{H}_t}$$

is a martingale measure for S (in the filtration $\mathcal{H}$) as soon as D is a positive martingale adapted to the filtration $\mathcal{W}$ such that $\mathbb{E}(D) = 1$.

3. Weak Information Modelling

We now turn to the weak approach. The main difference with the strong one is that there is no change of filtration but only a change a probability. Nevertheless, as it will be seen, from a mathematical point of view, all the results relative to initial enlargement of filtration can be recover from the weak approach.

The topic of investors with additional weak information was initiated in [5] and [6] and further studied in [8]. We present these works in a continuous setting.

We keep the notations of Section 1 and 2 and consider here an insider who is only weakly informed on Y. This means that the he has knowledge of the filtration $\mathcal{F}$ and of the law of Y. More precisely, with Y we associate a probability measure ν on $\mathcal{P}$. We assume that ν is equivalent to $\mathbb{P}_Y$ with a bounded density. The probability ν should be interpreted as the law of Y under the effective probability of the market.

In this section, we shall assume that the financial market

$$\left(\Omega, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}, (S_t)_{0 \leq t \leq T}\right)$$

is complete, i.e. that the process S enjoys the PRP.

3.1. **Conditioning of a Functional.**

3.1.1. *Minimal Probability Associated with a Conditioning.* We first associate with the weak information (Y,ν) a probability which will appear as canonical.

Definition 7. *The probability measure $\mathbb{P}^{\nu}$ defined on $(\Omega,\mathcal{F}_T)$ by:*

$$\mathbb{P}^{\nu}(A)=\int_{\mathcal{P}}\mathbb{P}(A\mid Y=y)\,\nu(dy)\,,\ \ A\in\mathcal{F}_T$$

is called the minimal probability associated with the weak information (Y,ν).

Here are some immediate consequences of this definition:

(1) If $F:\Omega\to\mathbb{R}$ is a bounded random variable then

$$\mathbb{E}^{\nu}(F\mid Y)=\mathbb{E}(F\mid Y)$$

(2) The law of Y under $\mathbb{P}^{\nu}$ is ν

(3) $\mathbb{P}^{\nu}=\mathbb{P}\Leftrightarrow\nu=\mathbb{P}_Y$

(4) The following equivalence relationship takes place

$$d\mathbb{P}^{\nu}=\frac{d\nu}{d\mathbb{P}_Y}(Y)\ \ d\mathbb{P}$$

(5) If $A\in\mathcal{F}_T$ is $\mathbb{P}-$independent of Y then it is also $\mathbb{P}^{\nu}-$independent of Y

(6) If we can choose a version of the map $y\to\mathbb{P}(\cdot\mid Y=y)$ which is continuous in the topology of weak convergence of probability measures, then the map $\nu\to\mathbb{P}^{\nu}$ is also continuous in this topology.

In order to justify the word *minimal* assigned to $\mathbb{P}^{\nu}$ we consider a convex function

$$\varphi:\mathbb{R}_+\to\mathbb{R}$$

and denote by $\mathcal{E}^{\nu}$ the set of probability measures on Ω which are equivalent to $\mathbb{P}$ and such that the law of Y under $\mathbb{Q}$ is ν.

Proposition 3. *(See* [5]*) We have*

$$\min_{\mathbb{Q}\in\mathcal{E}^{\nu}}\mathbb{E}\left(\varphi\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)=\mathbb{E}\left(\varphi\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)\right).$$

Proof. Let

$$d\mathbb{Q}=D\ d\mathbb{P}$$

be a probability measure which belongs to $\mathcal{E}^{\nu}$. Since the law of Y under $\mathbb{Q}$ is ν, we have

$$\mathbb{E}\left(D \mid Y\right) = \frac{d\nu}{d\mathbb{P}_Y}\left(Y\right).$$

Now, from Jensen's inequality

$$\varphi\left(\frac{d\nu}{d\mathbb{P}_Y}\left(Y\right)\right) \le \mathbb{E}\left(\varphi\left(D\right) \mid Y\right)$$

which implies

$$\mathbb{E}\left(\varphi\left(\frac{d\nu}{d\mathbb{P}_Y}\left(Y\right)\right)\right) \le \mathbb{E}\left(\varphi\left(D\right)\right).$$

□

Remark 11. *Notice that since $\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}$ is assumed to be bounded, the value $\mathbb{E}\left(\varphi\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)\right)$ is finite.*

Example 3. *With $\varphi\left(x\right) = x^2$, we see that $\mathbb{P}^{\nu}$ is the minimal variance probability, i.e.*

$$\inf_{\mathbb{Q}\in\mathcal{E}} \mathbb{E}\left(\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)^2\right) = \mathbb{E}\left(\left(\frac{d\mathbb{P}^{\nu}}{d\mathbb{P}}\right)^2\right).$$

With $\varphi\left(x\right) = x\ln x$, we see that it is also a minimal entropy measure.

3.1.2. *Semimartingale Decomposition under the Minimal Probability.* Our aim, now, is to develop stochastic calculus under the minimal probability $\mathbb{P}^{\nu}$. To do this, the first step is to compute the martingale density process of $\mathbb{P}^{\nu}$ with respect to $\mathbb{P}$. And then, we apply Girsanov's theorem. For this, we use the process $(\eta_t^y)_{0\le t<T, y\in\mathcal{P}}$ defined by Assumption 1 and the process $(\alpha_t^y)_{0\le t<T, y\in\mathcal{P}}$ defined in Lemma 1.

Lemma 4. *For $t < T$, $\mathbb{P}^{\nu}_{/\mathcal{F}_t}$ is absolutely continuous with respect to $\mathbb{P}_{/\mathcal{F}_t}$ and*

$$\mathbb{P}^{\nu}_{/\mathcal{F}_t} = \int_{\mathcal{P}} \eta_t^y \; \nu\left(dy\right) \; \mathbb{P}_{/\mathcal{F}_t}.$$

Proof. Since

$$\mathbb{P}^{\nu} = \frac{d\nu}{d\mathbb{P}_Y}\left(Y\right) \; \mathbb{P},$$

for $t < T$,

$$\mathbb{P}^{\nu}_{/\mathcal{F}_t} = \mathbb{E}\left(\frac{d\nu}{d\mathbb{P}_Y}\left(Y\right) \mid \mathcal{F}_t\right) \; \mathbb{P}_{/\mathcal{F}_t}.$$

Now, it is an immediate consequence of the definition of $(\eta_t^y)_{0\le t<T, y\in\mathcal{P}}$ that

$$\mathbb{E}\left(\frac{d\nu}{d\mathbb{P}_Y}\left(Y\right) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} \eta_t^y \; \nu\left(dy\right).$$

□

Theorem 11. *The process* $(S_t)_{0\leq t<T}$ *is a* $(\mathcal{F},\mathbb{P}^\nu)$ *semimartingale and its decomposition is given by*

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \frac{\int_{\mathcal{P}} \eta_u^y \alpha_u^y \nu(dy)}{\int_{\mathcal{P}} \eta_u^y \nu(dy)} + M_t, \quad t<T \tag{3.1}$$

where $(M_t)_{0\leq t<T}$ *is a* $(\mathcal{F},\mathbb{P}^\nu)$ *local martingale such that*

$$\langle M\rangle = \langle S\rangle.$$

Proof. The process

$$D_t = \int_{\mathcal{P}} \eta_t^y \ \nu(dy) \ , 0\leq t<T,$$

is the density process of $\mathbb{P}^\nu$ with respect to $\mathbb{P}$. By Lemma 1 and Fubini's theorem (we can apply it because of Assumption 2), we have for $0\leq t<T$, $1\leq i\leq d$,

$$d\langle D,S^i\rangle_t = \left(\int_{\mathcal{P}} \eta_t^y \left(\sum_{i=1}^d \alpha_t^{y,j} d\langle S^i,S^j\rangle_t\right)\nu(dy)\right).$$

The result is then a consequence of Girsanov's theorem. □

Remark 12. *The compensator of* S *under* $\mathbb{P}^\nu$*:*

$$\int_0^t d\langle S\rangle_u \frac{\int_{\mathcal{P}} \eta_u^y \alpha_u^y \nu(dy)}{\int_{\mathcal{P}} \eta_u^y \nu(dy)}$$

represents the information drift given by the weak anticipation ν*.*

3.1.3. *Connection with the Theory of Initial Enlargement of Filtration.* In this paragraph, we show the link between the weak and the strong approach, precisely the following theorem completes the converse of Jacod's theorem (see Theorem 2).

Theorem 12. *Under the probability* $\mathbb{P}^\nu$ *the price process* $(S_t)_{0\leq t<T}$ *is a semimartingale in the filtration* $\mathcal{G}$*, and its decomposition is given by*

$$S_t = s_0 + \int_0^t \langle S\rangle_u \alpha_u^Y + M_u, \qquad 0\leq t<T \tag{3.2}$$

where $(M_t)_{0\leq t<T}$ *is a* $\mathbb{P}^\nu$ *local martingale such that*

$$\langle S\rangle = \langle M\rangle.$$

Remark 13. *We recover the decomposition* (3.2) *from* (3.1) *with* $\nu=\delta_Y$*, nevertheless this is only formal because in our assumptions* ν *is not assumed to be random. This shows the analogy between Jacod's and Girsanov's theorem. This is very well explained in* [46]*, where the author understood the Jacod's theorem as a Girsanov formula applied on a convenient product probability space .*

Remark 14. *The decomposition* (3.1) *can also be written*

$$S_t = s_0 + \int_0^t d\langle S\rangle_u \mathbb{E}^\nu\left(\alpha_u^Y \mid \mathcal{F}_u\right) + M_t$$

which is a posteriori explained from the decomposition (3.2) *by the filtering theory.*

3.1.4. *Conditioned Stochastic Differential Equations.* We define here the notion of conditioned stochastic differential equations. This notion has been introduced in [5]. The idea is to construct a stochastic differential equation whose distribution of the solution is the same as the law of the process $(S_t)_{0\le t\le T}$ under $\mathbb{P}^\nu$. This gives hence a tool to construct *minimal* models associated with weak anticipations.

Let us assume that the information drift $\left(\frac{\int_{\mathcal{P}} \alpha_t^y \eta_t^y \nu(dy)}{\int_{\mathcal{P}} \eta_t^y \nu(dy)}\right)_{0\le t<T}$ is predictable with respect to the natural filtration of $(S_t)_{0\le t\le T}$, then there exists on the Wiener space of continuous functions $\mathbb{W}^d$ a predictable function F such that for all $t<T$,

$$\frac{\int_{\mathcal{P}} \alpha_t^y \eta_t^y \nu\left(dy\right)}{\int_{\mathcal{P}} \eta_t^y \nu\left(dy\right)} = F^\nu\left(t, (S_u)_{u\le t}\right).$$

Let us assume furthermore that under the martingale measure $\mathbb{P}$, $(S_t)_{0\le t\le T}$ can be written

$$S_t = s_0 + \int_0^t \sigma\left(s, (S_u)_{u\le s}\right) dW_s$$

where W is a standard Brownian motion and where σ is a predictable functional on the Wiener space valued in the space of $d\times d$ matrix.

Definition 8. *Let* $\left(\widetilde{\Omega}, (\mathcal{H}_t)_{0\le t<T}, \mathbb{Q}\right)$ *be any filtered probability space on which a d-dimensional standard* $\mathcal{H}-$*adapted Brownian motion* $(\beta_t)_{0\le t<T}$ *is defined. The stochastic differential equation*

$$X_t = s_0 + \int_0^t (\sigma^*\sigma)\left(s, (X_u)_{u\le s}\right) F^\nu\left(s, (X_u)_{u\le s}\right) ds + \int_0^t \sigma\left(s, (X_u)_{u\le s}\right) d\beta_s \tag{3.3}$$

$t<T$*, is called the conditioned stochastic differential equation (in abbreviate CSDE) associated with the conditioning* (T, Y, ν).

Remark 15. *By construction, the stochastic differential equation* (3.3) *has always a weak solution defined on the filtered probability space* $(\Omega, \mathcal{F}, \mathbb{P}^\nu)$.

Thanks to Yamada-Watanabe's theorem (see [43] pp. 368), we can now state an important transfer result:

Theorem 13. *Assume that the stochastic differential equation* (3.3) *enjoys the pathwise uniqueness property. Then* (3.3) *has a unique strong solution* $(X_t)_{0\leq t<T}$ *associated with the initial condition* $X_0 = s_0$ *and the law of* $(X_t)_{0\leq t<T}$ *is the same as the law of* $(S_t)_{0\leq t\leq T}$ *under the minimal probability* $\mathbb{P}^\nu$ *associated with the conditioning* (Y,ν).

3.1.5. *Connection with the Theory of Schrödinger processes.* Before we turn to examples of conditioning, in this paragraph we show how our results are closely related to some processes studied in quantum mechanics and called Schrödinger processes (see by e.g. [39]). We shall assume here that $(\Omega,\mathcal{F},\mathbb{P})$ is the d-dimensional Wiener space of continuous paths, i.e. that Ω is the space of continuous functions $[0,T]\to\mathbb{R}^d$ that $\mathcal{F}$ is the natural filtration of the coordinate process and that $\mathbb{P}$ is the Wiener measure. Hence, for a continuous stochastic process Z (defined on any suitable probability space), $Y(Z)$ denotes the $\mathcal{P}$-valued random variable $\omega\to Y((Z_t(\omega))_{0\leq t\leq T})$. Assume now that we are interested in the following variational problem:

Problem: *Let* ν *be a probability on* $\mathcal{P}$ *which is equivalent to* $\mathbb{P}_Y$ *with a bounded density. On a general filtered probability space*

$$\left(\tilde{\Omega},(\mathcal{H}_t)_{0\leq t\leq T},(W_t)_{0\leq t\leq T},\ \mathbb{Q}\right)$$

on which a standard Brownian motion W *is defined, we search an adapted control* Θ *minimizing the action*

$$\mathcal{A}=\frac{1}{2}\ \mathbb{E}^{\mathbb{Q}}\left(\int_0^T\Theta_s^2ds\right)$$

under the constraints

$$\mathbb{Q}\left(Y\left(Z^\Theta\right)\in dy\right)=\nu\left(dy\right),$$

and

$$\mathbb{P}_{Z^\Theta}\sim\mathbb{P}$$

where $\mathbb{P}_{Z^\Theta}$ *is the law of*

$$Z_t^\Theta=\int_0^t\Theta_sds+W_t,\ t\leq T.$$

For a probability

$$\tilde{\mathbb{Q}}=\mathcal{E}\left(\int_0^\cdot\Lambda_u\cdot dX_u\right)_T\ \mathbb{P}$$

on the Wiener space, the relative entropy of $\tilde{\mathbb{Q}}$ with respect to $\mathbb{P}$ (assuming that the integral below is convergent) is given by

$$\frac{1}{2}\ \mathbb{E}^{\tilde{\mathbb{Q}}}\left(\int_0^T\Lambda_u^2\ du\right).$$

Now, since the function $x \mapsto x\ln(x)$ is strictly convex on the open set $(0,\infty)$, Proposition 3 can be applied, and we conclude:

Theorem 14. *Assume that the relative entropy of ν with respect to $\mathbb{P}_Y$ is finite, then the previous problem admits one and only one solution Θ^* and the law $\mathbb{P}_{Z^{\Theta^*}}$ of the corresponding process Z^{Θ^*} satisfies the following absolute continuity relationship*

$$\mathbb{P}_{Z^{\Theta^*}} = \xi\left(Y\right)\ \mathbb{P}$$

where $\xi = \frac{d\nu}{d\mathbb{P}_Y}$.

Remark 16. *It is known from* [22] *that since*

$$\frac{1}{2}\ \mathbb{E}^{\mathbb{Q}}\left(\int_0^T \left(\Theta_s^*\right)^2 ds\right) < +\infty$$

the following limit (called Nelson forward derivative) exists in L^2

$$\mathcal{D}_t Z^{\Theta^*} := \lim_{h\rightarrow 0+} \mathbb{E}^{\mathbb{Q}}\left(\frac{Z_{t+h}^{\Theta^*} - Z_t^{\Theta^*}}{h} \mid \mathcal{F}_t\right),\ t<T$$

and is equal to Θ^*. This is hence by analogy with classical mechanics that

$$L_t = \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\ \left(\left(\mathcal{D}_t Z^{\Theta^*}\right)^2\right)$$

is called a Lagrangian and

$$\mathcal{A} = \int_0^T L_t\ dt$$

an action integral.

In the case where $Y = X_T$, X being the coordinate process, then the previous theorem is well known: Z^{Θ^*} is Markov and is called a Schrödinger process (for further details on this case, we refer to Subsection 3.2.2).

3.2. **Examples of conditioning.** We now give some examples of conditioning. These examples are the same as those studied in Subsection 2.2.

3.2.1. *Stochastic analysis.* Here again (see Section 2.2.1.), we use stochastic analysis to obtain explicit computations for the conditioning of a functional of the coordinate process. We assume that

$$\mathcal{P} = \mathbb{R}^N$$

for some integer $N \geq 1$ and that Y belongs to $\left(\mathbb{D}^{1,2}\right)^N$.

Proposition 4. *Assume that* $\xi := \frac{d\nu}{d\mathbb{P}_Y}$ *admits a continuously differentiable version with bounded partial derivatives then, for* $t < T$

$$\frac{\int_{\mathbb{R}^d} \eta_t^y \alpha_t^y \nu(dy)}{\int_{\mathbb{R}^d} \eta_t^y \nu(dy)} = \mathbf{D}_t \ln \mathbb{E}\left(\xi(Y) \mid \mathcal{F}_t\right) = \mathbb{E}^\nu\left(\left(\frac{\nabla \xi}{\xi}\right)^* (Y)\, \mathbf{D}_t Y \mid \mathcal{F}_t\right).$$

Proof. Under these assumptions, we have

$$\mathbb{P}^\nu = \xi(Y)\ \mathbb{P}$$

and so for $t \leq T$,

$$\mathbb{P}^\nu_{/\mathcal{F}_t} = \mathbb{E}\left(\xi(Y) \mid \mathcal{F}_t\right)\ \mathbb{P}_{/\mathcal{F}_t}.$$

Now from the Clark-Ocone formula (see [12], [37] pp. 183 and [40] Proposition 1.3.5.)

$$\begin{aligned}\mathbb{E}\left(\xi(Y) \mid \mathcal{F}_t\right) &= 1 + \textstyle\int_0^t \mathbb{E}\left(\mathbf{D}_s \xi(Y) \mid \mathcal{F}_s\right) \cdot dX_s \\ &= 1 + \textstyle\int_0^t \mathbb{E}\left((\nabla \xi(Y))^*\ \mathbf{D}_s Y \mid \mathcal{F}_s\right) \cdot dX_s.\end{aligned}$$

Hence,

$$\frac{\int_{\mathbb{R}^N} \alpha_t^y \eta_t^y \nu(dy)}{\int_{\mathbb{R}^N} \eta_t^y \nu(dy)} = \frac{\mathbb{E}\left((\nabla \xi(Y))^* \mathbf{D}_t Y \mid \mathcal{F}_t\right)}{\mathbb{E}\left(\xi(Y) \mid \mathcal{F}_t\right)}$$

and the Bayes formula gives the expected result. □

Remark 17. *Under the assumptions of the previous proposition, the formula for the compensator of* S *under* $\mathbb{P}^\nu$*, or the information drift takes hence the following nice form*

$$\mathbf{D}_t \ln \int \frac{\nu(dy)}{\mathbb{P}_Y(dy)} \mathbb{P}\left(Y \in dy \mid \mathcal{F}_t\right).$$

3.2.2. *Conditioning of the Terminal Value of a Diffusion.* Now, we return (see subsection 2.2.2) to the case $\mathcal{P} = \mathbb{R}^d$ and $Y = S_T$ where the dynamics of $(S_t)_{0 \leq t \leq T}$ under $\mathbb{P}$ are given by

$$S_t = s_0 + \int_0^t \operatorname{diag}(S_u)\, \sigma(S_u)\, dB_u, \qquad 0 \leq t \leq T \tag{3.4}$$

where $(B_t)_{0 \leq t \leq T}$ is a d-dimensional standard Brownian motion whose filtration is $\mathcal{F}$, $s_0 \in \left(\mathbb{R}_+^*\right)^d$ and σ a positive definite symmetric bounded C^∞ with bounded partial derivatives function such that

$$\inf_{x \in \mathbb{R}^d} \ \|\sigma \sigma^*(x)\| \geq a > 0.$$

And we consider again the transition function p_t, $0 < t \leq T$. In this setting, we easily obtain by the same way as in Section 2.2.2.

Theorem 15. *(See [8]) Under the probability $\mathbb{P}^{\nu}$, the process $(S_t)_{0\le t<T}$ admits the following semimartingale decomposition*

$$S_t = s_0 + \int_0^t (\tilde{\sigma}^*\tilde{\sigma})(S_u)\frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-u}(S_u,y)}{p_T(s_0,y)}\nu\,(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-u}(S_u,y)}{p_T(s_0,y)}\nu\,(dy)}du + \int_0^t \tilde{\sigma}\,(S_u)\,d\beta_u \tag{3.5}$$

where $(\beta_t)_{0\le t\le T}$ is a d-dimensional standard Brownian motion under $\mathbb{P}^{\nu}$ and

$$\tilde{\sigma}(x) = \mathrm{diag}\,(x)\,\sigma(x).$$

Remark 18. *Let us now consider on a filtered probability space*

$$\left(\tilde{\Omega}, (\mathcal{H}_t)_{0\le t\le T}\,, \mathbb{Q}\right)$$

which satisfies the usual conditions and on which is defined a d-dimensional standard Brownian motion $(W_t)_{0\le t\le T}$, the following stochastic differential equation

$$X_t = s_0 + \int_0^t (\tilde{\sigma}^*\tilde{\sigma})(X_u)\frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-u}(X_u,y)}{p_T(s_0,y)}\nu\,(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-u}(X_u,y)}{p_T(s_0,y)}\nu\,(dy)}du + \int_0^t \tilde{\sigma}\,(X_u)\,dW_u.$$

If this SDE enjoys the pathwise uniqueness property, then thanks to Theorem 13, it admits a strong solution whose law is the same as the law of S under $\mathbb{P}^{\nu}$. That is why we can speak of minimal model for the price process including the weak information ν on the price at the date T.

More generally, in [8], we have studied the conditioning of the value at time T of any Markov process. In this case, under the minimal probability measure, the price process is the so called Doob's transform of the starting process.

Precisely, here are the results. Let us denote by $\mathcal{L}$ the (extended) generator of S under $\mathbb{P}$. We assume that the domain $D(\mathcal{L})$ contains C^2 functions with compact support and that for such functions ϕ,

$$\mathcal{L}\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(x)\frac{\partial^2\phi}{\partial x_i \partial x_j} + \int_{\mathbb{R}^d\setminus\{0\}} \left(\phi(x+y) - \phi(x) - y\cdot\nabla\phi(x)\right) N(x,dy) \tag{3.6}$$

where $a(x)$ is a smooth function with values in nonnegative definite symmetric $d\times d$ matrices, and $N(x,dy)$ is the Lévy kernel of S (see [17]).

In this setting, there is a function d such that $\mathbb{E}[\xi(S_T)|\mathcal{F}_t] = d(t,S_t)$. Note that we assumed that ξ is a.s. bounded and strictly positive, so that $d(t,x)$ is strictly positive for almost all t and x.

Theorem 16. *(See [8]) $\mathbb{P}^\nu$ solves the martingale problem associated to $\mathcal{L}^\nu$ and the initial distribution δ_{s_0}, where*

$$\begin{aligned}\mathcal{L}^\nu\phi(t,x) &= \frac{1}{d(t,x)}\bar{\mathcal{L}}(\phi d)(t,x)\\ &= \frac{1}{d(t,x)}\frac{\partial(\phi d)}{\partial t}(t,x)+\frac{1}{d(t,x)}\mathcal{L}(\phi(t,.)d(t,.))(x)\end{aligned}$$

for any ϕ such that $\phi\in D(\bar{\mathcal{L}})$ and $\phi d\in D(\bar{\mathcal{L}})$, where $\bar{\mathcal{L}}$ is the generator of the space-time process (t,S_t) which is given by:

$$\bar{\mathcal{L}}\phi(t,x)=\frac{\partial\phi}{\partial t}(t,x)+\mathcal{L}(\phi(t,.))(x).$$

3.2.3. *Conditioning of the Perpetuity.* In this paragraph, we return to the case $\mathcal{P}=\mathbb{R}$ and assume that the dynamics of S under $\mathbb{P}$ is one-dimensional and given by

$$S_t=\exp\left(B_t-\frac{1}{2}t\right),\ t\geq 0$$

where $(B_t)_{0\leq t\leq T}$ is a one-dimensional standard Brownian motion whose filtration is $\mathcal{F}$. We have the following conditioning formula for the functional $\int_0^{+\infty}S_u^2du$.

Theorem 17. *(See [5], [7]) Assume that ν admits with respect to the function $\frac{e^{-\frac{1}{2y}}}{y^{3/2}}1_{y>0}$ a C^2 density $\xi:\mathbb{R}_+^*\to\mathbb{R}_+$ which is almost surely bounded. The decomposition of S under $\mathbb{P}^\nu$ is then given by*

$$S_t=1+\int_0^t S_v\left(1-2\frac{\int_0^{+\infty}e^{-u}u^{1/2}\xi\left(\int_0^v S_s^2ds+\frac{S_v^2}{2u}\right)du}{\int_0^{+\infty}e^{-u}u^{-1/2}\xi\left(\int_0^v S_s^2ds+\frac{S_v^2}{2u}\right)du}\right)dv+\beta_t\ ,\ t\geq 0 \tag{3.7}$$

where $(\beta_t)_{t\geq 0}$ is a standard Brownian motion under $\mathbb{P}^\nu$.

3.2.4. *Conditioning of Hitting Times.* Let $(B_t)_{0\leq t\leq T}$ be a one-dimensional standard Brownian motion whose filtration is $\mathcal{F}$. In this paragraph, we give the conditioning formula associated with

$$T_a=\inf\ \{t\geq 0\ ,\ B_t=a\}\ ,\ a>0.$$

Here, we assume that ν is a Borel measure defined on $\mathbb{R}_+^*$ by

$$\nu(dt)=\left(\int_0^{+\infty}e^{-t\frac{\delta^2}{2}+\delta a}\ m(d\delta)\ \right)\gamma(dt) \tag{3.8}$$

with m a probability measure on $\mathbb{R}_+^*$ such that $\int_0^{+\infty}\delta^2\ m(d\delta)<+\infty$ and

$$\gamma(dt)=\frac{a}{\sqrt{2\pi t^3}}\ e^{-\frac{a^2}{2t}}dt,\ t>0.$$

We recall that γ is the law of T_a under $\mathbb{P}$ (see [43] pp. 107).

Remark 19. *We take ν under the form* (3.8) *in order to use directly exponential martingales (see the expression* (3.9)*). Otherwise, we would have to invert the formula* (3.8) *, which appears to be rather complicated.*

We have then the following conditioning formula.

Theorem 18. *(See* [5]*) Under the probability $\mathbb{P}^{\nu}$ defined on $\mathcal{F}_{T_a}$ by*

$$\mathbb{P}^{\nu}_{/\mathcal{F}_{T_a}} = \int_0^{+\infty} e^{-T_a \frac{\delta^2}{2} + \delta a} \; m\,(d\delta) \;\; \mathbb{P}_{/\mathcal{F}_{T_a}}, \tag{3.9}$$

the process $(B_t)_{0 \le t \le T_a}$ is a semimartingale whose decomposition is given by

$$B_t = \int_0^t \frac{\int_0^{+\infty} \delta \; e^{-s\frac{\delta^2}{2} + \delta B_s} \; m\,(d\delta)}{\int_0^{+\infty} e^{-s\frac{\delta^2}{2} + \delta X_s} \; m\,(d\delta)} ds + \beta_t, \quad t \le T_a \tag{3.10}$$

where β is a standard Brownian motion under $\mathbb{P}^{\nu}$.

Example 4. *Let $\alpha > 0$. With $m = \frac{1}{2}\delta_{\alpha} + \frac{1}{2}\delta_{-\alpha}$, decomposition* (3.10) *becomes*

$$B_t = \alpha \int_0^t \tanh\left[\alpha B_s\right] ds + \beta_t.$$

3.3. **Pathwise Conditioning.** Before we conclude this section with some general comments on the conditioning, we would like to present briefly another kind of conditioning. Precisely, let us assume that an insider is in the following position: He knows that with probability one

$$\forall t > 0, \quad S_t \in \mathcal{O}$$

where $\mathcal{O} \subset \mathbb{R}^d$ is a non-empty, open, simply connected, and relatively compact set (the boundary of $\mathcal{O}$ shall be denoted by $\partial\mathcal{O}$).

Our question is now: Can we give to this insider a *minimal* model which takes into account this information ?

We shall answer this question in the case where, under the martingale measure $\mathbb{P}$, S is an homogeneous diffusion with elliptic generator $\mathcal{L}$ (the general case where S is only a local martingale seems to need more works).

For this, we proceed in two steps:

(1) First, we give a sense to the following probability measure

$$\mathbb{P}^* = \mathbb{P}(\cdot \mid \forall t > 0, S_t \in \mathcal{O})$$

(2) Secondly, we compute the semimartingale decomposition of S under $\mathbb{P}^*$.

To fulfill this program, we first define $\mathbb{P}^*$, as being the weak limit when $t \to +\infty$ (it is shown just below that it exists under suitable assumptions) of the following sequence of probabilities

$$\mathbb{P}^t = \mathbb{P}(\cdot \mid T_{\partial\mathcal{O}} = t), \quad t > 0$$

where

$$T_{\partial\mathcal{O}} = \inf\{t \leq 0,\ S_t \notin \mathcal{O}\}.$$

It is easily seen, by the strong Markov property of S under $\mathbb{P}$ that we have:

$$\mathbb{P}^t_{/\mathcal{F}_u} = \frac{g(S_u, t-u)}{g(s_0, t)}\ \mathbb{P}_{/\mathcal{F}_u}, \quad u < T_{\partial\mathcal{O}}$$

where g is defined by

$$\mathbb{P}(T_{\partial\mathcal{O}} \in dt \mid S_0 = x) = g(x,t)dt, \quad (x,t) \in \mathcal{O} \times \mathbb{R}^+.$$

Of course, implicitly, we assume that g such characterized is well defined and positive. Moreover, we shall assume that it is smooth.

Proposition 5. *We have, for all bounded and $\mathcal{F}_\infty$-measurable random variable F,*

$$\lim_{t \to +\infty} \mathbb{E}\left(\frac{g(S_u, t-u)}{g(s_0,t)}\ F\right) = \mathbb{E}\left(e^{\lambda_1 u}\psi_1(S_u)\ F\right)$$

where λ_1 is the smallest (positive) eigenvalue and ψ_1 the corresponding eigenfunction of the Dirichlet problem:

$$\frac{1}{2}\mathcal{L}\psi + \lambda\psi = 0 \ \textit{on}\ \mathcal{O},$$

and

$$\psi_{/\partial\mathcal{O}} = 0, \quad \psi_1(s_0) = 1.$$

Proof. In fact, this is a direct consequence of the theory of Dirichlet problems for elliptic operators on relatively compact open sets (these problems are widely discussed in [42]). Indeed, from this theory, g which solves the following problem:

$$\frac{\partial g}{\partial t} = \frac{1}{2}\mathcal{L}g \text{ on } \mathcal{O} \times \mathbb{R}^+ \text{ and } g(x,t) = \delta_0 \text{ on } \partial\mathcal{O} \times \mathbb{R}^+,$$

can be expanded on $\mathcal{O} \times \mathbb{R}^+$:

$$g(x,t) = \sum_{i=1}^{+\infty} e^{-\lambda_i t}\tilde{\psi}_i(x)$$

where $0 < \lambda_1 < ... < \lambda_n < ...$ are the eigenvalues and $\psi_1, ..., \psi_n, ...$ corresponding eigenfunctions of the Dirichlet problem

$$\frac{1}{2}\mathcal{L}\psi + \lambda\psi = 0 \text{ on } \mathcal{O}, \text{ and } \psi_{/\partial\mathcal{O}} = 0.$$

The proof follows then almost immediately after straightforward considerations. $\square$

From this, we deduce first that the following absolute continuity relationship holds:

$$\mathbb{P}^*_{/\mathcal{F}_u} = e^{\lambda_1 u}\psi_1(S_u)\ \mathbb{P}_{/\mathcal{F}_u}, \quad u < T_{\partial\mathcal{O}}$$

and secondly from Girsanov's theorem the semimartingale decomposition of S under $\mathbb{P}^*$ (up to $T_{\partial\mathcal{O}}$, but notice that actually $T_{\partial\mathcal{O}} = +\infty$, $\mathbb{P}^*$ a.s.).

Example 5. *As a consequence of this, we can deduce for instance that a one-dimensional standard Brownian motion B conditioned by the event $\{\forall t \geq 0, B_t \in [a,b]\}$ with $a < 0 < b$ is a Jacobi diffusion.*

Example 6. *A one-dimensional Brownian motion started from $a > 0$ and conditioned by the event $\{\forall t \geq 0, B_t > 0\}$ is a 3-dimensional Bessel process (which is clearly related to Pitman's theorem).*

3.4. **Comments.**

(1) In this section, we did not try to perform the most general setting in which our conditioning technique works. We preferred focussing on the ideas of the construction of $\mathbb{P}^\nu$. In fact Assumption 1 is even not necessary. Indeed, our conditioning technique stems of the semimartingale decomposition of the price process under the minimal probability $\mathbb{P}^\nu$. Now, since
$$\mathbb{P}^\nu = \frac{d\nu}{d\mathbb{P}_Y}(Y)\ \mathbb{P},$$
from Girsanov's theorem it is clear that the price process remains a semimartingale under $\mathbb{P}^\nu$ even if Assumption 1 is not satisfied. But this decomposition is not as explicit as the decomposition (3.1). We have worked in this section under this assumption in order to provide closed formulas and in order to show the link with the theory of initial enlargement of filtration.

(2) We can extend our work to the case where the measure ν is singular with respect to $\mathbb{P}_Y$. But for this, we need further assumptions. Indeed, the family of conditional probabilities $\mathbb{P}(\cdot \mid Y = y)$ is only defined $\mathbb{P}_Y$-a.s., so that the formula

$$\mathbb{P}^\nu = \int_{\mathcal{P}} \mathbb{P}(\cdot \mid Y = y)\,\nu(dy) \tag{3.11}$$

does not make sense when ν is singular with respect to $\mathbb{P}_Y$. Nevertheless, for instance, let us assume that we can choose a *canonical* version of the regular conditional distributions given Y such that the map $y \to \mathbb{P}(\cdot \mid Y = y)$ is continuous in the weak topology on probability measures. In this case, we can still define $\mathbb{P}^\nu$ by the formula 3.11, and it is not difficult to see that the law of Y under $\mathbb{P}^\nu$ is given by ν.

4. Utility Maximization with Weak Information

We now turn to financial applications of the conditioning. Precisely, we try to give a quantitative financial value to the weak information (Y, ν). This value should satisfy the following rule: Less the information is precise, less this value is. For instance the *minimal* information is $\nu = \mathbb{P}_Y$ because under $\mathbb{P}$ the price process $(S_t)_{0 \le t < T}$ is a local martingale and the *maximal* information is obtained at the limit with $\nu = \delta_y$ for $y \in \mathcal{P}$. Since the probability ν is assumed to be equivalent to $\mathbb{P}_Y$, we shall see that it implies that there is no arbitrage (in the sense defined Section 1); this is one of the main difference with the strong information setting. Throughout this section we shall denote by ξ a version of the density of ν with respect to $\mathbb{P}_Y$. Furthermore, we shall again assume here that ξ can be chosen bounded.

4.1. **Portfolio Optimization Problem.**

We first introduce the set of insiders which are weakly informed on the functional Y and define what will be called the financial value of the weak information (Y, ν). Precisely, let $\mathcal{E}^\nu$ be the set of probability measures $\mathbb{Q}$ on $(\Omega, \mathcal{F}_T)$ such that:

(1) $\mathbb{Q}$ is equivalent to $\mathbb{P}$
(2) $\mathbb{Q}\left(Y \in dy\right) = \nu\left(dy\right)$.

The financial market model associated with an element $\mathbb{Q}$ of $\mathcal{E}^\nu$ is

$$\left(\Omega, \left(\mathcal{F}_t\right)_{0 \le t < T}, \mathbb{Q}, (S_t)_{0 \le t \le T}\right). \tag{4.1}$$

It is then clear that there is no arbitrage on this market because $\mathbb{Q} \sim \mathbb{P}$ and S is a local martingale under $\mathbb{P}$. Now, the portfolio optimization problem associated with (4.1) is the following:

Portfolio optimization problem: *The insider's portfolio optimization problem is to find*

$$\sup_{\Theta \in \mathcal{A}_{\mathcal{F}}(S)} \mathbb{E}^{\mathbb{Q}}\left(U\left(x + \int_0^t \Theta_u dS_u\right)\right)$$

$x > 0$ *being the initial endowment of the insider.*

We recall here that U is a utility function and that $\mathcal{A}_{\mathcal{F}}(S)$ is the set of **adapted** admissible strategies (see Section 1).

Definition 9. *We define the financial value of the weak information (Y,ν) as being*

$$u(x,\nu) := \inf_{\mathbb{Q}\in\mathcal{E}^{\nu}} \sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}}\left(U\left(x+\int_0^T \Theta_u dS_u\right)\right).$$

In other terms, we define the financial value of the weak in formation as being the lowest increase in utility that can be gained by the insider from this extra knowledge .

4.1.1. *Value of the Weak Information in a Complete Market.* We assume in this subsection that the market is complete. The following proposition, which is just, by convex duality, a consequence of proposition 3 shows the universal property of $\mathbb{P}^{\nu}$ ($\mathbb{P}^{\nu}$ does not depend on the utility function used by the insider) among the other elements of $\mathcal{E}^{\nu}$. It also gives the exact value of the weak information. We use here classical results on martingale dual approach in a complete market (see [31] and [32]).

Theorem 19. *Assume that integrals below are convergent. Then for each initial investment $x>0$,*

$$\begin{aligned} u(x,\nu) &= \sup\nolimits_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\nu}\left(U\left(x+\textstyle\int_0^T \Theta_u dS_u\right)\right) \\ &= \int_{\mathcal{P}} (U\circ I)\left(\tfrac{\Lambda(x)}{\xi(y)}\right)\nu(dy) \end{aligned}$$

where $\Lambda(x)$ is defined by

$$\int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\mathbb{P}_Y(dy) = x.$$

Moreover, under $\mathbb{P}^{\nu}$ the optimal wealth process is given by

$$V_t = \int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\eta_t^y \mathbb{P}_Y(dy),$$

and the corresponding number of parts invested in the risky asset S by

$$\Theta_t = \int_{\mathcal{P}} I\left(\frac{\Lambda(x)}{\xi(y)}\right)\eta_t^y \alpha_t^y \mathbb{P}_Y(dy).$$

Proof. We will use a duality argument, as is now classical in this type of problem. Let $\mathbb{Q} = D\mathbb{P}\in\mathcal{E}^{\nu}$. We first proceed to rewrite

$$\sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}}\left(U\left(x+\int_0^T \Theta_u\cdot dS_u\right)\right)$$

differently. In fact, from classical results on complete markets, the Lagrangian associated with the optimization problem above is known to be given by

$$L(y) = xy + \mathbb{E}^{\mathbb{P}}\left(D\tilde{U}\left(\frac{y}{D}\right)\right) \qquad (y>0)$$

where we recall that $\tilde{U}$ is the convex conjuguate of U.

Moreover

$$\sup_{\Theta\in\mathcal{A}(S)}\mathbb{E}^{\mathbb{Q}}\left(U\left(x+\int_0^T\Theta_u\cdot dS_u\right)\right)=\inf_{y>0}L(y).$$

Hence we have

$$u(x,\nu)=\inf_{y>0}\left\{xy+\inf_D\mathbb{E}^{\mathbb{P}}\left(D\tilde{U}\left(\frac{y}{D}\right)\right)\right\}$$

where D runs through the densities $d\mathbb{Q}/d\mathbb{P}$, $\mathbb{Q}\in\mathcal{E}^\nu$. Now the function $z\mapsto z\tilde{U}\left(\frac{y}{z}\right)$ is convex for fixed y, and by proposition 3, we obtain

$$u(x,\nu)=\inf_{y>0}\left\{xy+\mathbb{E}^{\mathbb{P}}\left(\xi(Y)\tilde{U}\left(\frac{y}{\xi(Y)}\right)\right)\right\}.$$

The function $y\mapsto xy+\mathbb{E}^{\mathbb{P}}\left(\xi(Y)\tilde{U}\left(\frac{y}{\xi(Y)}\right)\right)$ inherits from $\tilde{U}$ the properties to be strictly convex, continuously differentiable and to tend to $+\infty$ as $y\to+\infty$; hence there exists $\Lambda(x)$ that realizes the inf, and $\Lambda(x)$ is given by

$$x-\mathbb{E}^{\mathbb{P}}\left(\xi(Y)I\left(\frac{\Lambda(x)}{\xi(Y)}\right)\right)=0.$$

From this proof, it also follows immediately that if we denote by V the optimal wealth process under $\mathbb{P}^\nu$ then we have:

$$V_T=I\left(\Lambda\left(x\right)\frac{d\mathbb{P}_Y}{d\nu}\left(Y\right)\right)$$

which implies the second part of the theorem. □

As in the case of a strong information on Y, we give explicitly the formulas for the most commonly used utility functions.

Example 7. *(See* [5], [8]*).*

(1) *Let* $\alpha\in(0,1)$ *and* $U\left(x\right)=\frac{x^\alpha}{\alpha}$ *then*

$$u\left(x,\nu\right)=\frac{x^\alpha}{\alpha}\left[\int_{\mathcal{P}}\left(\frac{d\nu}{d\mathbb{P}_Y}\left(y\right)\right)^{\frac{1}{1-\alpha}}\mathbb{P}\left(Y\in dy\right)\right]^{1-\alpha}$$

(2) *Let* $U\left(x\right)=\ln x$ *then*

$$u\left(x,\nu\right)=\ln x+\int_{\mathcal{P}}\frac{d\nu}{d\mathbb{P}_Y}\left(y\right)\ln\frac{d\nu}{d\mathbb{P}_Y}\left(y\right)\mathbb{P}\left(Y\in dy\right).$$

Remark 20. $u\left(x,\nu\right)$ *represents the minimal value of the terminal utility for an insider which is weakly informed on the functional* Y*. It will be shown later that we always have*

$$u\left(x,\nu\right)\geq U\left(x\right)$$

and that the equality takes place for $\nu = \mathbb{P}_Y$. Hence, it is perhaps more natural (but of course equivalent) to define the value of the additional information as being

$$v(x,\nu) := u(x,\nu) - U(x).$$

From an intuitive point of view, the more the anticipation is precise, the greater $v(x,\nu)$ will be. For instance, in the case of a logarithmic utility, $v(x,\nu)$ is the relative entropy of ν with respect to $\mathbb{P}_Y$.

Remark 21. *We have the following interesting additivity property: If an insider is weakly informed on two variables Y_1 and Y_2 which are independent under $\mathbb{P}$ then*

$$\mathcal{E}^{\nu_1 \otimes \nu_2} = \mathcal{E}^{\nu_1} \cap \mathcal{E}^{\nu_2}$$

and, again in the case of a logarithmic function

$$v(x,\nu_1 \otimes \nu_2) = v(x,\nu_1) + v(x,\nu_2).$$

4.1.2. *Value of the Weak Information in an Incomplete Market.* We now turn to the case where the market is incomplete, i.e. we assume that S does not enjoy the PRP. Moreover, we make the following additional assumption on the asymptotic elasticity of our utility function (see [34]):

$$\limsup_{x\to\infty} \frac{xU'(x)}{U(x)} < 1.$$

This last assumption allows to use the classical duality methods (see [32] and [34]).

Theorem 20. *(See [8]) For each initial investment $x > 0$,*

$$u(x,\nu) = \inf_{y>0} \left(\left(\inf_{\pi\in\mathcal{D}} \int_{\mathcal{P}} \widetilde{U}(y\pi(u))\,\nu(du) \right) + xy \right)$$

where

$$\mathcal{D} = \{ \frac{d\widetilde{\mathbb{P}}_Y}{d\nu},\ \widetilde{\mathbb{P}} \in \mathcal{M}(S) \}.$$

Proof. First fix $\mathbb{Q} \in \mathcal{E}^\nu$, and let

$$u_{\mathbb{Q}}(x,\nu) = \sup_{\Theta\in\mathcal{A}(S)} \mathbb{E}^{\mathbb{Q}} \left[U\left(x + \int_0^T \Theta_u \cdot dS_u \right) \right].$$

According to Theorem 2.2 in [34], we have

$$u_{\mathbb{Q}}(x,\nu) = \inf_{y>0} \left\{ xy + \inf_{\widetilde{\mathbb{P}}\in\mathcal{M}(S)} \mathbb{E}^{\mathbb{Q}} \left[\widetilde{U}\left(y\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}} \right) \right] \right\}.$$

We deduce that our function u is given by

$$u(x,\nu)=\inf_{\widetilde{\mathbb{P}}\in\mathcal{M}(S)}\inf_{y>0}\left\{xy+\inf_{\mathbb{Q}\in\mathcal{E}^{\nu}}\mathbb{E}^{\mathbb{Q}}\left[\widetilde{U}\left(y\frac{d\widetilde{\mathbb{P}}}{d\mathbb{Q}}\right)\right]\right\}.$$

Now, the optimal $\mathbb{Q}$ associated with a fixed $\widetilde{\mathbb{P}}$ is given by $\widetilde{\xi}(Y)\widetilde{\mathbb{P}}$, where $\widetilde{\xi}=\frac{d\nu}{d\widetilde{\mathbb{P}}_Y}$. Hence,

$$\begin{aligned}
u(x,\nu) &= \inf_{y>0}\inf_{\widetilde{\mathbb{P}}\in\mathcal{M}(S)}\left\{xy+\widetilde{\mathbb{E}}\left[\widetilde{\xi}(Y)\widetilde{U}\left(\frac{y}{\widetilde{\xi}(Y)}\right)\right]\right\}\\
&= \inf_{y>0}\inf_{\pi\in\mathcal{D}}\left\{xy+\widetilde{\mathbb{E}}\left[\frac{1}{\pi(Y)}\tilde{U}(y\pi(Y))\right]\right\}
\end{aligned}$$

and we conclude with straightforwards computations. □

Remark 22. *We always have*

$$u\left(x,\nu\right)\geq U\left(x\right).$$

Indeed, since $\widetilde{U}$ *is a convex function, for* $\xi\in\mathcal{D}$ *and* $y>0$*,*

$$\int_{\mathcal{P}}\widetilde{U}\left(y\xi\left(u\right)\right)\nu\left(du\right)\geq\widetilde{U}\left(y\right)$$

and hence,

$$u\left(x,\nu\right)\geq\inf_{y>0}\left(\widetilde{U}\left(y\right)+xy\right)=U\left(x\right).$$

Moreover, if $1\in\mathcal{D}$*, i.e. there exists* $\widetilde{\mathbb{P}}\in\mathcal{M}\left(S\right)$ *such that*

$$\widetilde{\mathbb{P}}\left(Y\in dy\right)=\nu\left(dy\right)$$

then

$$u\left(x,\nu\right)=U\left(x\right).$$

It would be really interesting to know more about the quantity

$$\inf_{\pi\in\mathcal{D}}\int_{\mathcal{P}}\widetilde{U}\left(y\pi\left(u\right)\right)\nu\left(du\right)$$

which seems to be hard to evaluate, even in the simplest examples of incomplete markets (Stochastic volatility models). Nevertheless, in the case $Y=S_T$, we can deduce from the previous proposition the following inequality.

Proposition 6. *Assume that* $\mathcal{P}=\mathbb{R}^d$ *and* $Y=S_T$*. Then we have for* $x>0$

$$u\left(x,\nu\right)\geq\int_{\left(\mathbb{R}_+^*\right)^d}U\left(\alpha+\beta\cdot u\right)\nu\left(du\right)$$

where $\alpha \in \mathbb{R}_+$ and $\beta \in \mathbb{R}_+^d$ are defined by

$$\frac{x \int_{\left(\mathbb{R}_+^*\right)^d} U'(\alpha+\beta\cdot u)\,\nu(du)}{\int_{\left(\mathbb{R}_+^*\right)^d} (\alpha+\beta\cdot u)\,U'(\alpha+\beta\cdot u)\,\nu(du)} = 1$$

$$\frac{x \int_{\left(\mathbb{R}_+^*\right)^d} u\ U'(\alpha+\beta\cdot u)\,\nu(du)}{\int_{\left(\mathbb{R}_+^*\right)^d} (\alpha+\beta\cdot u)\,U'(\alpha+\beta\cdot u)\,\nu(du)} = S_0.$$

Proof. Let $y>0$. Let us consider the convex functional

$$\begin{array}{rccl} L: & \mathcal{V} & \rightarrow & \mathbb{R}_+ \cup \{+\infty\} \\ & \xi & \rightarrow & \int_{\mathbb{R}_+^{*\,d}} \widetilde{U}\left(y\xi(u)\right)\nu(du) \end{array}$$

defined on the convex set

$$\mathcal{V} = \{\xi \geq 0,\ \int_{\mathbb{R}_+^{*\,d}} \xi(u)\,\nu(du) = 1,\ \int_{\mathbb{R}_+^{*\,d}} u\xi(u)\,\nu(du) = S_0\}.$$

Since $\mathcal{D} \subset \mathcal{V}$, we have

$$\inf_{\xi\in\mathcal{D}} \int_{\mathbb{R}_+^{*\,d}} \widetilde{U}\left(y\xi(u)\right)\nu(du) \geq \inf_{\xi\in\mathcal{V}} \int_{\mathbb{R}_+^{*\,d}} \widetilde{U}\left(y\xi(u)\right)\nu(du).$$

Now, because L is convex, in order to find the minimum of L on $\mathcal{V}$, it suffices to find a critical point. An easy computation shows that such a critical point $\widetilde{\xi}$ must satisfy

$$\int_{\mathbb{R}_+^{*\,d}} \widetilde{U}'\left(y\widetilde{\xi}(u)\right)\eta(u)\,\nu(du) = 0$$

for all η such that

$$\int_{\mathbb{R}_+^{*\,d}} \eta(u)\,\nu(du) = 0,\ \int_{\mathbb{R}_+^{*\,d}} u\eta(u)\,\nu(du) = 0.$$

This implies

$$\widetilde{\xi}(u) = \frac{1}{y} U'\left(\alpha(y) + \beta(y)\cdot u\right)$$

where $\alpha(y) \in \mathbb{R}_+$ and $\beta(y) \in \mathbb{R}_+^d$ are defined by

$$\int_{\mathbb{R}_+^{*\,d}} U'\left(\alpha(y)+\beta(y)\cdot u\right)\nu(du) = y$$

$$\int_{\mathbb{R}_+^{*\,d}} u\ U'\left(\alpha(y)+\beta(y)\cdot u\right)\nu(du) = y\ S_0$$

(see Remark 23 below). Hence

$$u(x,\nu) \geq \inf_{y>0}\left(\int_{\mathbb{R}_+^d} \widetilde{U}\left(U'\left(\alpha(y)+\beta(y)\cdot u\right)\right)\nu(du) + xy\right),$$

but for all $z>0$

$$\widetilde{U}\left(U'(z)\right) = U(z) - zU'(z),$$

hence

$$u(x,\nu) \geq \inf_{y>0} \left(\int_{\mathbb{R}_+^d} U(\alpha(y) + \beta(y) \cdot u) \, \nu(du) - (\alpha(y) + \beta(y) \cdot u) \, y + xy \right).$$

The previous infimum is clearly attained for $y > 0$ such that

$$\alpha(y) + \beta(y) \cdot S_0 = x$$

which gives the expected result. □

Remark 23. *$\alpha(y)$ and $\beta(y)$ are well defined by the formulas above. Indeed, according to a classical theorem of Hadamard, the application*

$$(\alpha, \beta) \mapsto \left(\int_{(\mathbb{R}_+^*)^d} U'(\alpha + \beta \cdot u) \, \nu(du), \int_{(\mathbb{R}_+^*)^d} u \, U'(\alpha + \beta \cdot u) \, \nu(du) \right)$$

induces a diffeomorphism from $(\mathbb{R}_+^* \times (\mathbb{R}_+^*)^d) \backslash (0,0)$ onto itself because it is proper with an everywhere non-singular differential. Moreover, since the function $y \mapsto \alpha(y) + \beta(y) \cdot S_0$ maps intervals into intervals, there exists y such that $\alpha(y) + \beta(y) \cdot S_0 = x$.

4.2. **Study of a Minimal Markov Market.** In this section, we make a complete study of the market

$$\left(\Omega, (\mathcal{F}_t)_{t \leq T}, (S_t)_{t \leq T}, \mathbb{P}^\nu \right)$$

in a Markov setting. Precisely, we consider the case where S is one-dimensional and $Y = S_T$ and we assume furthermore that there exist a bounded C^∞ function with bounded derivatives $\sigma : \mathbb{R}_+ \to [a, +\infty[$ $(a > 0)$ and a $\mathbb{P}$-standard Brownian motion $(B_t)_{0 \leq t \leq T}$ whose filtration is $\mathcal{F}$ such that

$$dS_t = S_t \sigma(S_t) \, dB_t \ , \ 0 \leq t \leq T. \tag{4.2}$$

From Theorem 15, we get:

Proposition 7. *Under the minimal probability $\mathbb{P}^\nu$,*

$$S_t = s_0 + \int_0^t S_u^2 \sigma(S_u)^2 \frac{\partial}{\partial x} \ln \varphi(u, S_u) \ du + \int_0^t S_u \sigma(S_u) \, d\beta_u \ , \ 0 \leq t \leq T \tag{4.3}$$

where β is a $(\mathcal{F}, \mathbb{P}^\nu)$ standard Brownian motion and φ the solution of the partial differential equation

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2} x^2 \sigma^2(x) \frac{\partial^2 \varphi}{\partial x^2} = 0 \tag{4.4}$$

associated with the limit condition

$$\varphi(T, x) = \xi(x).$$

After the study of the dynamics of the price process under $\mathbb{P}^{\nu}$, we now turn to optimal strategies. As it is shown in the next proposition, the optimal wealth process and the corresponding strategy associated with the model (4.3) are Markovian. We stress the fact that this is not the case in all generality, and that this property is once again characteristic of the minimal probability $\mathbb{P}^{\nu}$. In fact, the following proposition is a direct consequence of Theorem 19.

Proposition 8. *In the market*

$$\left(\Omega, (\mathcal{F}_t)_{t\leq T}, (S_t)_{t\leq T}, \mathbb{P}^{\nu}\right)$$

the optimal wealth process $(V_t)_{0\leq t\leq T}$ *is Markovian and can be written*

$$V_t = h\left(t, S_t\right)$$

where

$$h(t,y) = \mathbb{E}\left(I\left(\frac{\Lambda(x)}{\xi(S_T)}\right) \mid S_t = y\right).$$

Remark 24. *It is interesting to note that we can deduce from this that the optimal proportion process is given by*

$$\Pi_t\left(x\right) = S_t\ \pi\left(t, S_t\right)$$

where π *solves the partial differential equation*

$$\frac{\partial \pi}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x}\left(x^2\sigma^2\left(x\right)\frac{\partial \pi}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(x^2\sigma^2\left(x\right)\pi^2\right) = 0.$$

This equation, called the Burgers equation is well-known in fluid mechanics and particulary in aerodynamics (see [13]*).*

Let us now assume moreover that the volatility σ is a strictly positive constant.
Hence

$$dS_t = \sigma S_t dB_t$$

which implies

$$S_t = S_0 e^{\sigma B_t - \frac{\sigma^2}{2}t}$$

hence, by a change of variable, a weak information on S_T is equivalent to a weak information on the functional B_T, precisely

Proposition 9. *Under the minimal probability* $\mathbb{P}^{\nu}$,

$$dS_t = \sigma S_t dB_t\ ,\ t < T \tag{4.5}$$

where B satisfies

$$dB_t = \frac{\int_{-\infty}^{+\infty} \left(\frac{y-B_t}{T-t}\right) e^{\frac{y^2}{2T}-\frac{(y-B_t)^2}{2(T-t)}} \nu(dy)}{\int_{-\infty}^{+\infty} e^{\frac{y^2}{2T}-\frac{(y-B_t)^2}{2(T-t)}} \nu(dy)} dt + d\beta_t, \ t < T \tag{4.6}$$

β being a $(\mathcal{F}, \mathbb{P}^\nu)$ standard Brownian motion.

For a complete study of the stochastic differential equation (4.6) (which enjoys the pathwise uniqueness property), we refer to [5]. An immediate corollary of Proposition 8 is:

Proposition 10. *In the market*

$$\left(\Omega, (\mathcal{F}_t)_{t\le T}, (S_t)_{t\le T}, \mathbb{P}^\nu\right)$$

the optimal wealth process is given by

$$V_t = h(t, B_t)$$

where B is defined by (4.6) and h given by

$$h(t,y) = \frac{1}{\sqrt{2\pi(T-t)}} \int_{-\infty}^{+\infty} I\left(\frac{\Lambda(x)}{\xi(z)}\right) e^{-\frac{(z-y)^2}{2(T-t)}} \, dz.$$

We conclude this section with a particular case of the above market, precisely we study the case where ν is a Gaussian:

$$\nu(dx) = \frac{e^{-\frac{(x-m)^2}{2s^2}}}{\sqrt{2\pi}s} dx$$

with $m \in \mathbb{R}$ and $s^2 \le T$. In this case, B is a Gaussian process and then S a log-normal process. Precisely, straightforwards computations lead to

$$dS_t = \sigma S_t dB_t$$

where B satisfies

$$dB_t = \frac{\left(s^2 - T\right) B_t + mT}{\left(s^2 - T\right) t + T^2} dt + d\beta_t.$$

Remark 25. *For $s^2 = T$, we recover the Black and Scholes model*

$$\frac{dS_t}{S_t} = \mu dt + \sigma d\beta_t$$

with

$$\mu = \sigma \frac{m}{T}.$$

In this special and interesting case, all the computations can be made explicitely.

Proposition 11. *Let*

$$\delta = \frac{s^2 - T}{T}$$

the relative variance ($\delta = 0$ corresponds to the Black and Scholes model). The optimal proportion processes Π_t and the optimal expected utilities $u(x,\nu)$ for the utility functions U below are given as follows:

(1) *Logarithmic utility $U :]0,+\infty[\to \mathbb{R}$, $x \to \ln x$.*

$$\Pi_t = \frac{1}{\sigma}\frac{\delta B_t + m}{\delta t + T} \ , \ 0 \le t \le T,$$

$$u(x,\nu) = \ln x + \frac{1}{2}\left(\delta - \ln(1+\delta) + \frac{m^2}{T}\right).$$

(2) *Power utility $U :]0,+\infty[\to \mathbb{R}$, $x \to \frac{x^\alpha}{\alpha}$, $\alpha \in]0,1[$.*

$$\Pi_t = \frac{1}{\sigma}\frac{\delta B_t + m}{\delta t + T(1-\alpha) - \alpha\delta T} \ , \ 0 \le t \le T,$$

$$u(x,\nu) = \frac{x^\alpha}{\alpha}\ \frac{1}{\sqrt{1+\delta}}\left(\frac{1-\alpha}{\frac{1}{1+\delta} - \alpha}\right)^{\frac{1-\alpha}{2}} \exp\left(\frac{\alpha m^2}{2\left(T(1-\alpha) - \alpha\delta T\right)}\right).$$

5. Modelling of a Weak Information Flow

In this last section, we give a framework to model a weak information flow. Precisely, we study three cases:

(1) The insider has a knowledge of all the conditional laws $\mathbb{P}(Y \in dy \mid \mathcal{F}_t)$, $0 \le t < T$.
(2) The insider is allowed to update his weak information according to the information he receives.
(3) The insider is in the following position: At time t, he receives a weak information about the price S_{t+dt}, this anticipation being only valid for the infinitesimal time dt and at $t + dt$ the investor receives some new information and makes a new anticipation, and so on....

5.1. **Dynamic Conditioning.** In this section, we consider an insider who knows all the conditional laws of Y. Precisely, with Y we associate a continuous $\mathcal{F}$-adapted process $(\nu_t)_{0\le t\le T}$ of probability measures on $\mathcal{P}$ (assumed to be the conditional laws of Y under the effective probability of the market) such that $\nu_T = \delta_Y$.

We will assume that for $0 \leq t < T$, ν_t admits an almost surely strictly positive bounded density ξ_t with respect to $\mathbb{P}(Y \in dy \mid \mathcal{F}_t)$, i.e.

$$\nu_t(dy) = \xi_t(y)\,\mathbb{P}(Y \in dy \mid \mathcal{F}_t).$$

Remark 26. *The terminal condition $\nu_T = \delta_Y$ is equivalent to $\xi_T = 1$.*

Let $\mathcal{E}^\nu$ be the set of probability measures $\mathbb{Q}$ on Ω such that:

(1) $\mathbb{Q}$ is equivalent to $\mathbb{P}$

(2) $\mathbb{Q}(Y \in dy \mid \mathcal{F}_t) = \nu_t(dy)$, $t < T$.

In order to ensure that $\mathcal{E}^\nu$ is non empty we have to make the following additional assumption on ν_t.

Assumption 3. *There exists a $\mathcal{P}(\mathcal{F}) \otimes \mathcal{B}(\mathcal{P})$ measurable process $(\lambda_t^y)_{0 \leq t < T}$ and an adapted d-dimensional semimartingale $(A_t)_{0 \leq t < T}$ such that:*

(1) *For $\mathbb{P}_Y - a.e.$ $y \in \mathcal{P}$ and for $0 \leq t < T$, $1 \leq i, j \leq d$*
$$\mathbb{E}\left(\int_0^t \left(\lambda_u^{y,i}\right)^2 d\langle A^i, A^j\rangle_u\right) < +\infty$$

(2)
$$\nu_t(dy) = \nu_0(dy) + \int_0^t \left(\lambda_s^y \mathbb{P}_Y(dy)\right) dA_s,\ t < T$$

(3)
$$\nu_t \to_{t \to T}^{weakly} \delta_Y.$$

In order to understand this, let us give a simple example of a sequence $(\nu_t)_{0 \leq t < T}$ which satisfies this assumption. Assume that $(S_t)_{0 \leq t \leq T}$ is a d-dimensional standard Brownian motion and that

$$\nu_t(dy) = q_{T-t}(S_t, y)dy$$

where $(q_t)_{0 \leq t < T}$ is the transition function of the diffusion with generator

$$\mathcal{L} = b(x)\nabla + \frac{1}{2}\Delta$$

where b is a bounded smooth function whose all partial derivatives are also bounded.
In this case, it easily seen (by Itô's formula) that our assumption holds with

$$\lambda_u^y = \sqrt{2\pi T}\ e^{\frac{y^2}{2T}}\ \frac{\partial q_{T-u}}{\partial x}(S_u, y)$$

and

$$A_t = S_t - \int_0^t b(S_u)du.$$

Notice that in this case

$$\mathcal{E}^{\nu} = \{\mathbb{Q}\}$$

where $\mathbb{Q}$ is the probability equivalent to $\mathbb{P}$ such that

$$S_t - \int_0^t b(S_u)du$$

is a martingale under $\mathbb{Q}$.

The following proposition characterizes the set $\mathcal{E}^{\nu}$. Before we state it, recall that a probability measure valued process $(\nu_t)_{0\leq t<T}$ is called a martingale, if for any bounded and measurable function f, the process $\int_{\mathcal{P}} f(y)\nu_t(dy)$ is a martingale.

Proposition 12. *Let $\mathbb{Q} = D\ \mathbb{P}$ a probability measure on Ω equivalent to $\mathbb{P}$, then the following assertions are equivalent:*

i) $\mathbb{Q} \in \mathcal{E}^{\nu}$

ii) The probability measure valued process $(\nu_t)_{0\leq t\leq T}$ is a $\mathcal{F}$ martingale under $\mathbb{Q}$

iii) The process $(D_t\xi_t(Y))_{0\leq t\leq T}$ is a $\mathcal{G}$ martingale under $\mathbb{P}$, where $D_t = \mathbb{E}(D \mid \mathcal{F}_t)$ and where $\mathcal{G}$ is the initial enlargement of $\mathcal{F}$ by Y

iv) The process $(A_t)_{0\leq t<T}$ is a $\mathcal{F}$ martingale under $\mathbb{Q}$.

Proof. i) $\Rightarrow$ ii)

Let us consider $\mathbb{Q} \in \mathcal{E}^{\nu}$.

Then for all bounded and measurable function f

$$\mathbb{E}^{\mathbb{Q}}\left(f(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} f(y)\,\nu_t(dy)\,,\ t \leq T,$$

which implies immediately ii).

ii) $\Rightarrow$ iii)

If $(\nu_t)_{0\leq t\leq T}$ is a $\mathcal{F}-$martingale under $\mathbb{Q}$ then

$$\mathbb{E}\left(D \mid \mathcal{G}_t\right) = \xi_t(Y)\,\mathbb{E}\left(D \mid \mathcal{F}_t\right),\ t \leq T.$$

Indeed, for all bounded and measurable function f,

$$\mathbb{E}^{\mathbb{Q}}\left(f(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} f(y)\,\nu_t(dy)\,,\ t \leq T$$

thus,

$$\mathbb{E}\left(\frac{D}{D_t}\,f(Y) \mid \mathcal{F}_t\right) = \int_{\mathcal{P}} f(y)\,\xi_t(y)\,\mathbb{P}\left(Y \in dy \mid \mathcal{F}_t\right),\ t \leq T$$

and so,

$$\mathbb{E}\left(\frac{D}{D_t}\, f(Y) \mid \mathcal{F}_t\right) = \mathbb{E}\left(f(Y)\,\xi_t(Y) \mid \mathcal{F}_t\right),\ t \leq T.$$

This means that for all bounded and $\mathcal{F}_t-$ measurable functional F

$$\mathbb{E}\left(\frac{D}{D_t}\, f(Y)\, F\right) = \mathbb{E}\left(f(Y)\,\xi_t(Y)\, F\right),\ t \leq T$$

which provides

$$\mathbb{E}\left(D \mid \mathcal{G}_t\right) = \xi_t(Y)\, D_t,\ t \leq T,$$

so that the process $(D_t\xi_t(Y))_{0\leq t\leq T}$ is a $\mathcal{G}$ martingale under $\mathbb{P}$.

iii) $\Rightarrow$ i)

In this case, we have

$$\mathbb{E}\left(D \mid \mathcal{G}_t\right) = \xi_t(Y)\, \mathbb{E}\left(D \mid \mathcal{F}_t\right),\ t \leq T$$

which implies that for all bounded and measurable function f,

$$\begin{array}{rcl}\mathbb{E}^{\mathbb{Q}}\left(f(Y) \mid \mathcal{F}_t\right) & = & \mathbb{E}\left(f(Y)\,\frac{\mathbb{E}(D\mid\mathcal{G}_t)}{D_t} \mid \mathcal{F}_t\right) \\ & = & \mathbb{E}\left(f(Y)\,\xi_t(Y) \mid \mathcal{F}_t\right) \\ & = & \int_{\mathcal{P}} f(y)\,\xi_t(y)\,\mathbb{P}\left(Y \in dy \mid \mathcal{F}_t\right) \\ & = & \int_{\mathcal{P}} f(y)\,\nu_t(dy).\end{array}$$

iv)$\Leftrightarrow$ii)

Immediate. □

The set $\mathcal{E}^{\nu}$ is hence the set of martingale measures for A.

5.2. **Dynamic Correction of a Weak Information.**

5.2.1. *A Useful Convergence Lemma for SDE's.* Before, we define our framework for a dynamic correction of the weak information, we state a lemma about the continuity (with respect to a suitable norm) of solutions of stochastic differential equations with respect to the drift. This lemma will be used in the following.

Let us consider a sequence of Borel functions

$$a_n : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d,\ n \in \mathbb{N}$$

and let us also consider a Borel function $\sigma : [0,T] \times \mathbb{R}^d \to \mathcal{M}_d(\mathbb{R})$, where $\mathcal{M}_d(\mathbb{R})$ is the space of $d \times d$ matrix.

We make the following assumptions. There exist non negative constants K_1 and K_2 such that for $(t,x,y) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\left\|a_n(t,x) - a_n(t,y)\right\| + \left\|\sigma(t,x) - \sigma(t,y)\right\| \leq K_1 \left\|x - y\right\|$$

and

$$\|a_n(t,x)\|^2 + \|\sigma(t,x)\|^2 \le K_2\left(1+\|x\|^2\right).$$

Furthermore, we assume that the following pointwise convergence holds

$$a_n \to_{n\to+\infty} a$$

where $a:[0,T]\times\mathbb{R}^d \to \mathbb{R}^d$ is a Borel function. Notice that we do not assume the continuity with respect to t of the functions a_n.

Now we consider on a filtered probability space

$$\left(\Omega, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P}\right)$$

which satisfies the usual conditions and on which a standard d-dimensional Brownian motion is defined, the sequence X^n, where $(X_t^n)_{0\le t\le T}$ is the solution of the stochastic differential equation

$$X_t^n = x_0 + \int_0^t a_n(s, X_s^n)\,ds + \int_0^t \sigma(s, X_s^n)\,dW_s.$$

We can note that the assumptions made on a_n and σ ensure the existence and the uniqueness of such a solution. We have the following theorem.

Theorem 21. *Under the above conditions, we have*

$$\lim_{n\to+\infty} \mathbb{E}\left(\sup_{0\le t\le T}\|X_t^n - X_t\|^2\right) = 0$$

where $(X_t)_{0\le t\le T}$ is the solution of the stochastic differential equation

$$X_t = x_0 + \int_0^t a(s, X_s)\,ds + \int_0^t \sigma(s, X_s)\,dW_s.$$

Proof. For notational convenience, we make the proof in dimension $d=1$ but it immediately extends to the d dimensional case.

Since for $x, y \in \mathbb{R}^2$

$$(x+y)^2 \le 2\left(x^2+y^2\right) \tag{5.1}$$

we have for $t\in[0,T]$ and $n\in\mathbb{N}$

$$\begin{aligned}(X_t^n - X_t)^2 \quad \le \quad & 2\left[\int_0^t \left(a_n(u, X_u^n) - a(u, X_u)\right)du\right]^2 \\ & +2\left[\int_0^t \left(\sigma(u, X_u^n) - \sigma(u, X_u)\right)dW_u\right]^2.\end{aligned}$$

Now, from Cauchy-Schwarz inequality and (5.1)

$$\begin{aligned}\left[\int_0^t \left(a_n(u, X_u^n) - a(u, X_u)\right)du\right]^2 \quad \le \quad & 2T\int_0^t \left(a_n(u, X_u^n) - a_n(u, X_u)\right)^2 du \\ & +2T\int_0^t \left(a_n(u, X_u) - a(u, X_u)\right)^2 du.\end{aligned}$$

Thus,

$$\begin{aligned}\left[\int_0^t \left(a_n\left(u, X_u^n\right) - a\left(u, X_u\right)\right) du\right]^2 &\leq 2TK_1^2 \int_0^t \left(X_u^n - X_u\right)^2 du \\ &\quad +2T \int_0^t \left(a_n\left(u, X_u\right) - a\left(u, X_u\right)\right)^2 du.\end{aligned}$$

On the other hand, from Burkholder-Davis-Gundy inequality, for $0 \leq t \leq \tau$

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left[\int_0^t \left(\sigma\left(u, X_u^n\right) - \sigma\left(u, X_u\right)\right) dW_u\right]^2\right) \leq 4K_1^2 \mathbb{E}\left(\int_0^\tau \left(X_u^n - X_u\right)^2 du\right).$$

Putting things together, we deduce the following estimation

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left(X_t^n - X_t\right)^2\right) &\leq (2T+8) K_1^2 \mathbb{E}\left(\int_0^\tau \left(X_u^n - X_u\right)^2 du\right) \\ &\quad +2T\mathbb{E}\left(\int_0^t \left(a_n\left(u, X_u\right) - a\left(u, X_u\right)\right)^2 du\right).\end{aligned} \tag{5.2}$$

We apply now Gronwall's lemma to obtain

$$\begin{aligned}\mathbb{E}\left(\left(X_\tau^n - X_\tau\right)^2\right) &\leq 2T(2T+8) K_1^2 e^{(2T+8)K_1^2\tau} \int_0^\tau e^{-(2T+8)K_1^2 u} G_n(u)\, du \\ &\quad +2TG_n(\tau)\end{aligned}$$

where

$$G_n(\tau) := \mathbb{E}\left(\int_0^\tau \left(a_n\left(u, X_u\right) - a\left(u, X_u\right)\right)^2 du\right).$$

The uniform linear growth assumption on a_n allows to use the dominated convergence theorem which ensures first the following pointwise convergence

$$G_n \rightarrow_{n\rightarrow+\infty} 0$$

A new use of the dominated convergence theorem in (5.2) gives the expected result, i.e. for $0 \leq \tau \leq T$

$$\mathbb{E}\left(\sup_{0\leq t\leq\tau}\left(X_t^n - X_t\right)^2\right) \rightarrow_{n\rightarrow+\infty} 0$$

This complete the proof. □

5.2.2. *Dynamic Correction of the Weak Information Flow.* In this section, we define a framework for a dynamic correction of the weak information at each time t. The insider is allowed to update his weak information according to the information he receives. The reasoning here is more pathwise oriented in the spirit of the notion of minimal model (see subsection 3.1.4).

Here we shall assume that the dynamics of $(S_t)_{0\leq t\leq T}$ under the martingale measure $\mathbb{P}$ are given by

$$S_t = s_0 + \int_0^t \operatorname{diag}\left(S_u\right) \sigma\left(S_u\right) dW_u, \qquad 0 \leq t \leq T$$

where $(W_t)_{0\le t\le T}$ is a d-dimensional standard Brownian motion, $s_0 \in \left(\mathbb{R}_+^*\right)^d$ and σ a positive definite symmetric bounded C^∞ function with bounded partial derivatives function satisfying

$$\inf_{x\in\mathbb{R}^d} \quad \|\sigma\sigma^*(x)\| \ge a > 0.$$

Let us consider a sequence of subdivisions

$$\mathcal{S}_n = \{0 \le t_0 < ... < t_i < ... < t_n = T\}, \quad n \in \mathbb{N}^*$$

of the time interval $[0,T]$, whose mesh tends to 0 when $n \to +\infty$.

The idea now is to associate with S_T an $\sigma(S_{t_i})$–adapted random sequence $(\nu_{t_i})_{i=0,...,n-1}$ of probability measures on $\mathcal{P}$ corresponding to an updating of the weak information on S_T at time t_i. This updating can come from the observation of the prices in the time interval $[0,t_i]$ as well as the learning of a new information on S_T. Let us now try to construct a model for the price process which takes into account these updatings. We shall furthermore assume that the insider has no other informations on the price process.

At time t_i the insider "learns" ν_{t_i} (which **erases completely** $\nu_{t_{i-1}}$) and constructs a probabilistic bridge in the time interval $[t_i,T]$ which condition S_T to follow conditionally to the past filtration $\mathcal{F}_{t_i}$ the law ν_{t_i}. Since this anticipation is only valid in the time interval $[t_i,t_{i+1})$, according to Theorem 15 and Remark 18, the model is given by

$$S_u - S_{t_i} = \int_{t_i}^u (\tilde\sigma^*\tilde\sigma)(S_v) \frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)}\nu_{t_i}(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)}\nu_{t_i}(dy)} dv + \int_{t_i}^u \tilde\sigma(S_v)\, dW_v, t_i \le u < t_{i+1}$$

where $(W_t)_{0\le t\le T}$ is a d-dimensional standard Brownian motion and $\tilde\sigma(x) = \operatorname{diag}(x)\,\sigma(x)$.

Putting things together, we obtain the following dynamics

$$S_t = s_0 + \int_0^t (\tilde\sigma^*\tilde\sigma)(S_v)\, a_n(v,S_v)\, dv + \int_0^t \tilde\sigma(S_v)\, dW_v, 0 \le t \le T$$

where

$$a_n(v,S_v) = \sum_{i=0}^{n-1} \left(\frac{\int_{\mathbb{R}^d} \frac{\nabla p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)}\nu_{t_i}(dy)}{\int_{\mathbb{R}^d} \frac{p_{T-v}(S_v,y)}{p_{T-t_i}\left(S_{t_i},y\right)}\nu_{t_i}(dy)} \right) 1_{[t_i,t_{i+1})}(v).$$

Notice now that the following pointwise convergence holds

$$a_n \to_{n\to+\infty} a$$

where

$$a_n(t,x) = \int_{\mathbb{R}^d} \frac{\nabla p_{T-t}(S_t,y)}{p_{T-t}(S_t,y)}\nu_t(dy).$$

Hence, if we are under the assumptions of Theorem 21, then

$$S_t = s_0 + \int_0^t (\tilde{\sigma}^* \tilde{\sigma})(S_u) \int_{\mathbb{R}^d} \frac{\nabla p_{T-u}}{p_{T-u}} (S_u, y)\, \nu_u(dy)\, du + \int_0^t \tilde{\sigma}(S_u)\, dW_u, 0 \le t \le T \tag{5.3}$$

is a model for the price process which takes into account the dynamic correction of the weak information flow.

5.2.3. *An example of dynamic correction.* Let us now assume that the dynamics of $(S_t)_{0 \le t \le T}$ under $\mathbb{P}$ is given by

$$dS_t = \sigma S_t dB_t, 0 \le t \le T$$

where $(B_t)_{0 \le t \le T}$ is a standard Brownian motion whose filtration is $\mathcal{F}$ and σ a strictly positive constant. With this model, we associated the two parameter Gaussian diffusion

$$dB_t = \frac{\left(s^2 - T\right) B_t + mT}{(s^2 - T)\, t + T^2} dt + d\beta_t \tag{5.4}$$

m being a mean parameter and s a variance parameter (we recall that $B_T \sim \mathcal{N}\left(m, s^2\right)$). This model is the model which has been introduced at the end of section 4.2.

With this two parameters model, it is natural to make the maximum likelihood method on the time interval $[0, t]$, $t > 0$, in order to check our anticipation. This gives after some straightforward computations the following estimators

$$m_t = \frac{T}{t} B_t,$$

and

$$s_t^2 = 0.$$

Hence, we apply formally the dynamic correction procedure, we obtain thanks to (5.3)

$$dB_t = \frac{B_t}{t} dt + dW_t$$

which is a singular equation (Precisely, if B is a standard Brownian motion then the natural filtration of

$$\left(B_t - \int_0^t \frac{B_s}{s} ds \right)_{t<T}$$

completed by $\sigma(B_T)$ is the natural filtration of $(B_t)_{0 \le t \le T}$, see [38] for further details). This helps understand, that without any exogeneous information on B_T (in this case ν_t is constructed only by observing the past of a Brownian motion), the dynamic correction leads to a singularity.

Nevertheless, for a general process $(\nu_t)_{0\leq t<T}$, the dynamic correction procedure applied to the model (5.4) , gives according to 5.3

$$dB_t = \frac{\int_{\mathbb{R}} y \ \nu_t (dy) - B_t}{T-t} \ dt + d\tilde{W}_t.$$

This equation implies something interesting, indeed it implies

$$\mathbb{E}(B_t) = (T-t)\int_0^t \frac{\mathbb{E}(\mu_s)}{(T-s)^2} ds, \ t<T$$

with $\mu_t = \int_{\mathbb{R}} y \ \nu_t (dy)$. This corresponds to the natural intuition that at each time $t<T$, the tangent to the curve $s \to \mathbb{E}(B_s)$ hits the line $s=T$ at the point $(T, \mathbb{E}^*(\mu_t))$, a well-known phenomenon in physics which is related to the notion of caustic.

5.3. **Dynamic Information Arrival.** Now, we would like to present in this short section a nice construction. We will not be rigorous in order to make understand the main intuitions.

Here again, we consider the case where S is one-dimensional and assume furthermore that there exist a bounded C^∞ function $\sigma : \mathbb{R}_+ \to [a, +\infty[$ $(a>0)$ and a $\mathbb{P}$-standard Brownian motion $(B_t)_{0\leq t\leq T}$ whose filtration is $\mathcal{F}$ such that

$$dS_t = S_t \sigma(S_t) \, dB_t, \quad 0 \leq t \leq T.$$

Let us consider an insider who is in the following position: At time t, he receives a weak information about the price S_{t+dt}, this anticipation being only valid for the infinitesimal time dt and at $t+dt$ the investor receives some new information and makes a new anticipation, and so on....

Such an insider tries hence to construct a probability measure $\mathbb{P}^*$ on Ω such that

$$\mathbb{P}^*(S_{t+dt} \in dy \mid \mathcal{F}_t) = \xi_t(y) \, \mathbb{P}(S_{t+dt} \in dy \mid \mathcal{F}_t)$$

$y \to \xi_t(y)$ being the $\mathcal{F}_t-$measurable function, corresponding to the weak information that the investor receives at time t.

A canonical way to construct $\mathbb{P}^*$ is the following : At time t, he learns the weak information (S_{t+dt}, ξ_t) and constructs in the sense of Section 2 a probabilistic bridge in the time interval $[t, t+dt)$ which forces S_{t+dt} to follow conditionally to the past filtration $\mathcal{F}_t$ (which is not trivial contrary to the static case) the law $\nu_t(dy) = \xi_t(y) \, \mathbb{P}(S_{t+dt} \in dy \mid \mathcal{F}_t)$. Let us see what it implies on $\mathbb{P}^*$.

Let us denote by $\widetilde{\sigma}$ the function $x \to x\sigma(x)$. It is easily seen, from a representation theorem, that we have

$$\xi_t(S_{t+dt}) = 1 + \xi_t'(S_t)\,\widetilde{\sigma}(S_t)\,dB_t$$

(assumed that ξ_t is differentiable) because of the normalization

$$\mathbb{E}(\xi_t(S_{t+dt}) \mid \mathcal{F}_t) = 1$$

Hence, for all test function f, we must have thanks to Itô's formula (notice that $\xi_t(S_t) = 1$)

$$\begin{aligned} \mathbb{E}^*(f(S_{t+dt}) \mid \mathcal{F}_t) &= \mathbb{E}(\xi_t(S_{t+dt})\,f(S_{t+dt}) \mid \mathcal{F}_t) \\ &= f(S_t) + f'(S_t)\,\xi_t'(S_t)\,\widetilde{\sigma}(S_t)^2\,dt + \tfrac{1}{2}f''(S_t)\,\widetilde{\sigma}^2(S_t)\,dt \end{aligned}$$

which implies that the semimartingale decomposition of S under $\mathbb{P}^*$ is

$$dS_t = \xi_t'(S_t)\,\widetilde{\sigma}(S_t)^2\,dt + \widetilde{\sigma}(S_t)\,d\beta_t$$

where β is a standard Brownian motion under $\mathbb{P}^*$.

6. Comments

It seems that there are many advantages to the use of the weak approach instead of the strong one in the modelling of informations on financial markets. The first reason is the robustness of this kind of modelling. Mathematically, it means that the map $\nu \to \mathbb{P}^\nu$ is continuous in the weak topology (as soon as we have a continuous version for the maps $y \to \mathbb{P}(A \mid Y = y)$). In practice, it means that a little modification on the parameters of the model does not change completely the model. Of course this robustness does not hold for initial enlargement of filtration. The second reason is practical. Indeed, the weak approach generates simple models which are easy to calibrate and to implement numerically. And finally, last but not least, the third reason is theoretical. Indeed, it has been seen that the theory of initial enlargement of filtration can be deduced from the weak approach by taking formally for ν some anticipative measures (precisely $\nu = \delta_Y$ gives the enlargement $\mathcal{F} \vee \sigma(Y)$).

To conclude, it is important to stress the fact that more works need to be done.

(1) On one hand, it would be very interesting to study more thoroughly some examples of incomplete markets, such as stochastic volatility models.

(2) On the other hand, it would also be interesting to apply the present approach to equilibrium price models in the spirit of [4] and [35]; for instance, given two agents acting in the same market S with different weak anticipations, what is the equilibrium price of S?

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