Math 545
Homework no. 3

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**Problem 1.** Let \( f \in L^p(\mathbb{R}^n) \), \( 1 \leq p \leq \infty \), \( g \in L^1(\mathbb{R}^n) \). Prove that \( f \ast g \in L^p(\mathbb{R}^n) \), and that moreover
\[
\|f \ast g\|_{L^p(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.
\]

**Problem 2.** Let \( E \subset \mathbb{R}^n \) be measurable and \( \Omega \subset \mathbb{R}^m \) be an open set, and define for a measurable function \( f : E \times \Omega \to \mathbb{R} \)
\[
F(y) \overset{\text{def}}{=} \int_E f(x, y) \, dx , \quad y \in \Omega.
\]
Assume that \( f \), \( \partial f / \partial y_j \in C(E \times \Omega, \mathbb{R}) \), \( j = 1, \ldots, m \). Suppose furthermore that there exist functions \( \phi, \psi_j \in L^1(E) \) such that for \( x \in E \) and \( y \in \Omega \) one has
\[
|f(x, y)| \leq \phi(x) , \quad \left| \frac{\partial f}{\partial y_j}(x, y) \right| \leq \psi_j(x) , \quad j = 1, \ldots, m.
\]
Prove that under such hypothesis one has \( F \in C^1(\Omega) \), and for every \( j = 1, \ldots, m \)
\[
\frac{\partial F}{\partial y_j}(y) = \int_E \frac{\partial f}{\partial y_j}(x, y) \, dx .
\]

**Problem 3.** Consider in \( \mathbb{R}^3 \) the function
\[
F(\xi) \overset{\text{def}}{=} \int_{S^2} e^{-i \langle \xi, \omega \rangle} \, d\sigma(\omega) , \quad \xi \in \mathbb{R}^3.
\]

a) Determine the range of \( p \in [1, \infty] \) such that \( F \in L^p(\mathbb{R}^3) \).
b) Determine the range of \( a > 0 \) for which the function \( F(\xi) / |\xi|^a \) is Riemann integrable in the improper sense in \( \mathbb{R}^3 \). By this we mean that there exists finite
\[
\lim_{\epsilon \to 0, R \to \infty} \int_{\epsilon < |\xi| < R} \frac{F(\xi)}{|\xi|^a} \, d\xi .
\]

**Problem 4.** For \( \lambda > 0 \) define
\[
F(\xi) \overset{\text{def}}{=} \int_{S^{n-1}} e^{i \sqrt{\lambda} \langle \xi, \omega \rangle} \, d\sigma(\omega) , \quad \xi \in \mathbb{R}^n.
\]
a) Prove that \( F \in C^\infty(\mathbb{R}^n) \).

b) Prove that \( F \) is an eigenfunction of the Laplacian with corresponding eigenvalue \( \lambda \), i.e., \( F \) solves the equation in \( \mathbb{R}^n \)

\[
\Delta F = -\lambda F.
\]

c) Prove that \( F \) is spherically symmetric.

**Problem 5.** Let \( f \in C_0^\infty(\mathbb{R}) \). Prove that for any \( k \in \mathbb{N} \) there exists \( C = C(f, k) > 0 \), such that for \( \xi \in \mathbb{R} \setminus \{0\} \) one has

\[
|\hat{f}(\xi)| \leq \frac{C}{|\xi|^k}.
\]