Problem 1. Let $\Omega \subset \mathbb{R}^n$ be an open set. A linear functional $T : C^\infty_0(\Omega) \to \mathbb{C}$ is said to be a distribution on $\Omega$ if for every compact set $K \subset \Omega$ there exist $C \geq 0$ and $N \in \mathbb{N}_0$ such that
\[
| \langle T, \phi \rangle | \leq C \sum_{|\alpha| \leq N} \sup_{x \in K} |\partial^\alpha \phi(x)|
\]
for every $\phi \in C^\infty_0(\Omega)$ such that $\text{supp} \phi \subset K$. Denote by $\mathcal{D}'(\Omega)$ the space of distributions on $\Omega$. Prove that $T \in \mathcal{D}'(\Omega)$ if and only if
\[
\langle T, \phi_j \rangle \to 0 \quad \text{in} \quad \mathbb{C}
\]
for every sequence $\{\phi_j\}_{j \in \mathbb{N}}$ in $C^\infty_0(\Omega)$ such that:
\begin{enumerate}
\item There exists a compact set $K \subset \Omega$ such that $\text{supp} \, \phi_j \subset K$ for every $j \in \mathbb{N}$;
\item For every $\alpha \in \mathbb{N}_0^n$, one has $\partial^\alpha \phi_j \to 0$ uniformly as $j \to \infty$.
\end{enumerate}

Problem 2. Prove that $\mathcal{S}(\mathbb{R}^n) \subset L^p(\mathbb{R}^n)$ for $1 \leq p \leq \infty$.

Problem 3. a) Prove that if $T \in \mathcal{D}'(\Omega)$, then for any $\alpha \in \mathbb{N}_0^n$ one has $\partial^\alpha T \in \mathcal{D}'(\Omega)$.

b) Let $H$ be the Heaviside function on $\mathbb{R}$ defined by $H(x) = 1$ for $x \geq 0$, $H(x) = 0$ for $x < 0$. Compute $\frac{dH}{dx}$ in $\mathcal{D}'(\mathbb{R})$.

Problem 4. Prove that $\mathcal{S}(\mathbb{R}^n)$ is complete.

Problem 5. Prove that
\[
d(\phi, \psi) = \sum_{p=0}^{\infty} \frac{1}{2^p} \frac{||\phi - \psi||_p}{1 + ||\phi - \psi||_p}
\]
defines a distance on $\mathcal{S}(\mathbb{R}^n)$, and that the metric topology is equivalent with that originally given on $\mathcal{S}(\mathbb{R}^n)$.