ON THE BEST POSSIBLE CHARACTER OF THE $L^q$ NORM IN SOME A PRIORI ESTIMATES FOR NON-DIVERGENCE FORM EQUATIONS IN CARNOT GROUPS

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1. Introduction

In the recent paper [DGN2] we formulated the following conjecture. Consider a stratified, nilpotent Lie group $G$ of step $r$, with Lie algebra $\mathfrak{g} = V_1 \oplus \cdots \oplus V_r$. This means that $[V_j, V_j] = V_{j+1}$ for $j = 1, \ldots, r - 1$, whereas $[V_1, V_r] = \{0\}$. If $m = \dim V_1$, let $X_1, \ldots, X_m$ be a fixed orthonormal basis of the first layer $V_1$. Continue to denote with $X_1, \ldots, X_m$ the corresponding system of left-invariant vector fields on $G$. Given a $m \times m$ matrix-valued function $f : A(g) = (a_{ij}(g))$ on $G$ having measurable entries, suppose in addition that $A(g) = A(g)^T$, and that there exist $\nu > 0$ such that

$$\nu |\zeta|^2 \leq \sum_{i,j=1}^m a_{ij}(g) \zeta_i \zeta_j \leq \nu^{-1} |\zeta|^2,$$

for a. e. $g \in G$, and for every $\zeta \in \mathbb{R}^m$. For a function $u : G \to \mathbb{R}$ we introduce the symmetrized horizontal Hessian of $u$ as the $m \times m$ matrix-valued function $f : (u_{ij}(g))_{i,j=1}^m$ defined by

$$u_{ij} \overset{df}{=} \frac{1}{2} \left\{ X_i X_j u + X_j X_i u \right\}, \quad i, j = 1, \ldots, m.$$

The stratification of the Lie algebra carries a natural family of non-isotropic dilations $\Delta_\lambda \xi = \lambda \xi_1 + \lambda^2 \xi_2 + \cdots + \lambda^r \xi_r$, for any $\xi = \xi_1 + \xi_2 + \cdots + \xi_r \in \mathfrak{g}$. Such dilations are transferred to $G$ by means of the exponential map as follows $\delta_\lambda = \exp \circ \Delta_\lambda \circ \exp^{-1}$. We denote by $dg$ the bi-invariant Haar measure on $G$ obtained by pushing forward Lebesgue measure on $\mathfrak{g}$ via the
exponential map. One has \( d(g \circ \delta x) = \lambda^Q dg \). For this reason the number \( Q = \sum_{j=1}^{r} \text{dim} \, V_j \) is called the homogeneous dimension of \( G \).

**Conjecture:** Given a connected, bounded open set \( \Omega \subset G \), let \( F \in L^Q(\Omega) \). Suppose that \( u \in L^2(Q) \cap C(\overline{\Omega}) \) satisfy

\[
Lu \overset{\text{def}}{=} \sum_{i,j=1}^{m} a_{ij} \, u_{ij} \geq F
\]

in \( \Omega \). There exists a constant \( C = C(G, \nu, \Omega) > 0 \) such that

\[
\sup_{\Omega} u \leq \sup_{\partial \Omega} u^+ \ + \ C \, \| F \|_{L^Q(\Omega)}.
\]

Here, \( L^2(Q) \) indicates the Sobolev space of functions \( u \in L^2(\Omega) \) having weak derivatives \( X_i X_j u \in L^2(\Omega) \). We note explicitly that, thanks to the Sobolev type embedding in [F2], we have \( L^2(\Omega) \subset C(\Omega) \). In a standard fashion, an important consequence of (1.2) would be the uniqueness for the Dirichlet problem

\[
\left\{
\begin{array}{ll}
Lu = F & u \in L^2_{\text{loc}}(\Omega) \cap C(\overline{\Omega}), \\
\phi & \phi \in C(\partial \Omega).
\end{array}
\right.
\]

In the abelian case, when \( G = \mathbb{R}^m \) with the ordinary isotropic dilations, one has \( g = V_1 = \mathbb{R}^m \), so that \( Q = m = n \), and \( (u,ij) = D^2 u \), the standard Hessian matrix of \( u \). The above conjecture, in this situation, is in fact the celebrated geometric maximum principle of Alexandrov-Bakelman-Pucci (ABP, henceforth), see [A1], [Ba1], [Pu2], and also [Ba2], [Ba3]. In this setting it is well-known that an estimate such as (1.2) can only hold with the \( L^n \) norm of \( F \) in the right-hand side, in the sense that it is in general impossible to have it with any smaller norm, see Alexandrov [A2] and Pucci [Pu1].

In the sub-Riemannian setting, however, there is an unsettling discrepancy between the two geometric parameters involved, namely the dimension \( m \) of the first layer of the Lie algebra, and the homogeneous dimension \( Q \) of the group. The former plays a pervasive role in the intrinsic notion of convexity introduced in [DGN2], since such notion involves a quantitative control of \( u \) only on the horizontal subbundle of planes generated by the basis \( X_1, ..., X_m \) of \( V_1 \). Furthermore, \( m \) is also the correct dimension for the horizontal Monge-Ampère equation

\[
\det (u_{ij}) = F
\]

associated with the geometric estimate (1.2) [DGN3]. One has in fact from the classical geometric-arithmetic mean inequality

\[
\det (u_{ij}) \leq \frac{1}{m^m} \left[ \sum_{i,j=1}^{m} a_{ij} \, u_{ij} \right]^{m} \overset{\text{def}}{=} \det (a_{ij}).
\]

On the other hand, the fundamental solution of the horizontal Laplacian corresponding to the basis \( \{ X_1, ..., X_m \} \)

\[
Lu = tr (u_{ij}) = \sum_{j=1}^{m} X_j^2 u
\]
is homogeneous of degree $2 - Q$ with respect to the non-isotropic group dilations \( \{ \delta_\lambda \}_{\lambda > 0} \), see \([F^2]\).

This implies, see (4.3) in Section 4, that when \( a_{ij} = \delta_{ij} \), \( i, j = 1, \ldots, m \), one can considerably improve the conjectured estimate. In this case, in fact, (1.2) holds with the \( L^Q \) norm of \( F \) replaced by \( \| F \|_{L^P(\Omega)} \), where \( p > Q/2 \). Such result continues to be valid provided that the Green function \( G(g, g') \) for the operator \( L \) and the domain \( \Omega \) be controlled near the singularity by the fundamental solution of the horizontal Laplacian.

In view of these considerations it is not a priori clear whether in (1.2) the \( L^Q \) norm is optimal, or rather a smaller norm is permissible. The purpose of this note is to clarify this point, and prove the best possible character of the \( L^Q \) norm. Precisely, we will establish the following.

**Theorem 1.1.** Let \( G \) be a group of Heisenberg type, with homogeneous dimension \( Q \), and denote by \( \Omega \) the gauge ball \( B(e, 1) \), where \( e \) is the group identity. For every \( 0 < \epsilon < Q \) there exists a (real) matrix valued function \( A^\epsilon(g) = (a^\epsilon_{ij}(g)) \) with symmetric and bounded measurable entries, and satisfying (1.1) for some \( \nu_\epsilon > 0 \), such that the Dirichlet problem

\[
\begin{aligned}
L^\epsilon u &= \sum_{i,j=1}^m a^\epsilon_{ij} u_{ij} = 0 &\text{in}\quad \Omega, \\
uu &= 0 &\text{on}\quad \partial \Omega,
\end{aligned}
\]

admits a solution \( u = u^\epsilon \in L^{2Q-\epsilon}(\Omega) \cap C(\overline{\Omega}) \) different from the trivial one. As a consequence, we cannot replace the \( L^Q(\Omega) \) norm in the right-hand side of (1.2) with the smaller one in \( L^{2Q-\epsilon}(\Omega) \).

Theorem 1.1 is analogous to the well-known results of Alexandrov [A2] and Pucci [Pu1] for elliptic equations. Our proof is inspired to Serrin’s note [Se] on the existence of pathological solutions for divergence form operators. The main new fact here, and this is also the reason why in Theorem 1.1 we confine our attention to groups of Heisenberg type, is that our construction exploits crucially a deep property of such groups, namely that the non-isotropic gauge is a fundamental solution of the following fully nonlinear operator

\[
L_\infty \overset{\text{def}}{=} \sum_{i,j=1}^m u_{ij} X_i u X_j u.
\]

The role of this \( \infty \)-horizontal Laplacian will appear clearly in the course of the proof of Theorem 1.1. We mention that the solutions of the operator \( L_\infty \) can be viewed as the asymptotics as \( p \to \infty \) of solutions of the one parameter family of quasi-linear equations

\[
L_p u = \sum_{j=1}^m X_j (|X u|^{p-2} X_j u) = 0, \quad 1 < p < \infty.
\]

2. Preliminaries

The definition of a Carnot group has been given in the introduction. Since Carnot groups of step 2 will play a special role in this note, we briefly recall some additional notions and definitions which will be of use in the sequel. For more details the reader is referred to the paper
Given a Carnot group $G$ of step 2 with Lie algebra $\mathfrak{g} = V_1 \oplus V_2$, we assume that a scalar product $\langle \cdot, \cdot \rangle$ is given on $\mathfrak{g}$ for which the $V_i$'s are mutually orthogonal. The notation $\{X_1, \ldots, X_m\}$ will indicate a fixed orthonormal basis of the first layer $V_1$. We denote by $\{Y_1, \ldots, Y_k\}$ an orthonormal basis of the second layer $V_2$. Elements of $V_i$, $i = 1, 2$ are assigned the formal degree $i$. We use the letters $g, g'$ for points in $G$, whereas the letters $\xi, \xi'$ will indicate elements of the Lie algebra $\mathfrak{g}$. We will denote by $L_{g_0}(g) = g_0 g$ the left-translations on $G$ by an element $g_0 \in G$. Recall that the exponential map $\exp : \mathfrak{g} \to G$ is a global analytic diffeomorphism [V]. Denote by $\xi : G \to \mathfrak{g}$ its inverse. We define analytic maps $\xi_i : G \to V_i$, $i = 1, 2$, by letting $\xi_i = \pi_i \circ \xi$, where $\pi_i : \mathfrak{g} \to V_i$ denotes the projection onto the $i$-th layer. Clearly, $\xi(g) = \xi_1(g) + \xi_2(g)$. We will indicate with

\begin{equation}
(2.1) \quad x_j(g) = \langle \xi_1(g), X_j \rangle, \quad j = 1, \ldots, m, \quad y_s(g) = \langle \xi_2(g), Y_s \rangle, \quad s = 1, \ldots, k.
\end{equation}

the projections of the exponential coordinates of $g$ onto $V_1$ and $V_2$. Letting $x(g) = (x_1(g), \ldots, x_m(g))$, $y(g) = (y_1(g), \ldots, y_k(g))$, we will often identify $g \in G$ with its exponential coordinates $(x(g), y(g))$. Such identification is justified by the fundamental Baker-Campbell-Hausdorff formula [V]

\[ \exp \xi \exp \xi' = \exp (\xi + \xi' + \frac{1}{2} [\xi, \xi']) \text{, } \xi, \xi' \in \mathfrak{g}. \]

We notice that

\begin{equation}
(2.2) \quad |\xi_1(g)|^2 = |x(g)|^2 = \sum_{j=1}^{m} x_j(g)^2 \text{, } \quad |\xi_2(g)|^2 = |y(g)|^2 = \sum_{i=1}^{k} y_i(g)^2 \text{.}
\end{equation}

We continue to denote by $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_k$ the corresponding systems of left-invariant vector fields on $G$ defined by $X_j(g) = (L_g)_*(X_j)$, $Y_s(g) = (L_g)_*(Y_s)$, where $(L_g)_*$ denotes the differential of $L_g$. The system $X_1, \ldots, X_m$ defines a basis for the so-called horizontal subbundle $HG$ of the tangent bundle $TG$. The horizontal gradient of $u$ is defined by $Xu = X_1uX_1 + \ldots + X_muX_m$, and $|Xu| = (\sum_{j=1}^{m} (X_ju)^2)^{1/2}$.

The natural non-isotropic gauge is defined by

\begin{equation}
(2.3) \quad N(g) = (|x(g)|^4 + 16 |y(g)|^2)^{1/2} \text{.}
\end{equation}

Such gauge introduces a suitable pseudo-metric structure in $G$ by the formula $d(g, g') = N(g^{-1}g')$. Pseudo-balls with respect to $d$ are denoted by $B(g, r)$. We note that by left-translation invariance $B(g, r) = L_g(B(e, r))$, where $e \in G$ indicates the identity. We note explicitly that the (otherwise immaterial) factor 16 has been introduced in (2.3) for normalization purposes. Its role will be better explained by (3.4) in Section 3.

3. Groups of Heisenberg type, $\infty$-horizontal Laplacian and the gauge

In this section we analyze some basic geometric properties of Carnot groups of step 2, with special attention to those ambients which are relevant to the proof of Theorem 1.1. Let $G$ be a
group of step 2 with Lie algebra \( g = V_1 \oplus V_2 \). We define the Kaplan mapping \( J : V_2 \to \text{End}(V_1) \) by the equation

\[
< J(\eta)\xi_1, \xi_2 > = < \xi_1, \xi_2 > , \quad \xi_1, \xi_2 \in V_1 , \ \eta \in V_2 .
\]

It is clear from (3.1) that for every \( \eta \in V_2 \), and \( \xi \in V_1 \) one has \( < J(\eta)\xi, \xi > = 0 \). Our first result is a basic orthogonality property noted in [GV2], see also [DGN1].

**Lemma 3.1.** In a Carnot group \( G \) of step 2, consider the function \( \psi(g) \overset{\text{def}}{=} |x(g)|^2 \). One has \( X\psi = 2\xi_1 \), and for any \( s = 1, \ldots, k \),

\[
< X\psi, Xy_s > \equiv 0 , \quad < X\psi, X(|y|^2) > = 0 .
\]

As a consequence,

\[
< \xi_1, X(|y|^2) > = 0 .
\]

**Definition 3.2.** A Carnot group \( G \) of step 2 is called of Heisenberg type if for every \( \eta \in V_2 \), such that \( |\eta| = 1 \), the map \( J(\eta) : V_1 \to V_1 \) is orthogonal.

Definition 3.2 is due to A. Kaplan [K], who introduced such class of Lie groups in connection with questions of hypoellipticity. It was later recognized that there exists in nature a very rich supply of groups of Heisenberg type. For instance, it was proved in [CDKR] that the nilpotent component \( N \) in the Iwasawa decomposition of a simple group of rank one \( KAN \), is a group of Heisenberg type. Definition 3.2 implies

\[
|J(\eta)\xi| = |\eta| \ |\xi| , \quad < J(\eta')\xi, J(\eta'')\xi > = < \eta', \eta'' > |\xi|^2 , \quad \eta, \eta', \eta'' \in V_2 , \ \xi \in V_1 .
\]

A striking property of groups of Heisenberg type is that for an appropriate constant \( C = C(G) > 0 \) the function

\[
\Gamma(g, g') = \frac{C}{N(g^{-1}g')^{q-2}} ,
\]

satisfies the equation \( \mathcal{L}\Gamma(g, \cdot) = -\delta_g \) in \( \mathcal{D}'(G) \). This result generalized Folland’s fundamental solution for the Heisenberg group \( \mathbb{H}^n \) [F1]. We emphasize here that in a group of Heisenberg type the gauge (2.3) defines a true distance [Cy], not just a pseudo-distance as for a general Carnot group.

When the dimension of the second layer is \( k = 1 \), then a group of Heisenberg type is isomorphic to the Heisenberg group \( \mathbb{H}^n \), see [S]. In such case there is essentially one Kaplan mapping \( J \), given by the symplectic matrix in \( \mathbb{R}^{2n} \)

\[
S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}
\]

The next lemma expresses some basic symmetry properties of groups of Heisenberg type, see [K], [CDG], [GV2].
Lemma 3.3. Let $G$ be a group of Heisenberg type. One has

\begin{equation}
|XN(g)|^2 = \frac{|x(g)|^2}{N(g)^2}.
\end{equation}

\begin{equation}
|X(y_s)(g)|^2 = \frac{1}{4} |x(g)|^2, \quad s = 1, \ldots, k.
\end{equation}

\begin{equation}
|X(|y|^2)(g)|^2 = |x(g)|^2 |y(g)|^2.
\end{equation}

In [CDG] it was discovered that, remarkably, an appropriate power of the gauge (2.3) also provides a fundamental solution of the quasilinear operator $L_p$ in (1.7). It was proved there that for every $1 < p < \infty$, $p \neq Q$, there exists $C_p = C(G, p) > 0$ such that the function

\begin{equation}
\Gamma_p(g, g') = \frac{p-1}{p-Q} \frac{1}{\sigma_p^{-1/(p-1)} N(g^{-1} g')^{(Q-p)/(p-1)}}
\end{equation}

solves the equation $L_p \Gamma_p(g, \cdot) = -\delta_g$ in an appropriate weak sense. In (3.8) we have let

\begin{equation}
\sigma_p \overset{\text{def}}{=} Q, \quad \omega_p \overset{\text{def}}{=} Q \int_{B(e, 1)} |XN(g)|^p \, dg.
\end{equation}

We stress that the one-parameter family of quasilinear operators $L_p$ formally approaches in the limit as $p \to \infty$ the $\infty$-horizontal Laplacian defined by (1.6). What we mean by this is that

\[ L_p u = (p - 2) |Xu|^{p-4} \left\{ \frac{1}{p-2} |Xu|^2 \Delta u + \mathcal{L}_\infty u \right\}. \]

If $u_p$ is a solution to $L_p u_p = 0$, then the previous equation gives

\[ \frac{1}{p-2} |Xu_p|^2 \Delta u_p + \mathcal{L}_\infty u_p = 0. \]

If we assume that $u_p \to u$, and that $|Xu_p|^2 \Delta u_p$ is bounded independently of $p$ large, then by letting $p \to \infty$ we find that $u$ must be a solution to $\mathcal{L}_\infty u = 0$.

This being said, we return to (3.9) to note that

\[ \lim_{p \to \infty} \frac{\Gamma_p(g, g')}{\sigma_p^{-1/(p-1)}} = \frac{\|XN\|_{L^\infty(B(e, 1))}}{\|XN\|_{L^\infty(B(e, 1))}}, \]

therefore, for every $g, g' \in G, g \neq g'$, we have

\begin{equation}
\lim_{p \to \infty} \Gamma_p(g, g') = \frac{N(g^{-1} g')}{\|XN\|_{L^\infty(B(e, 1))}}.
\end{equation}

This remarkable formula suggests that the gauge $N$ should be a fundamental solution for the fully nonlinear operator (1.6). This intuition is confirmed by the following property of groups of Heisenberg type which constitutes the essential tool in the proof of Theorem 1.1.

Theorem 3.4. Let $G$ be a group of Heisenberg type, then one has in the classical sense

\[ \mathcal{L}_\infty N = 0, \quad \text{in} \ G \setminus \{e\}. \]

Proof. We observe the following alternative form of (1.6)

\begin{equation}
\mathcal{L}_\infty u = \frac{1}{2} < X(|Xu|^2), Xu >.
\end{equation}
Using (3.11) we find
\[ \mathcal{L}_\infty N = \frac{1}{2} < X(|XN|^2), XN > = \frac{1}{2} < X(N^{-2}|x|^2), XN > , \]
where in the latter equation we have used (3.5) in Lemma 3.3. We now have, again by (3.5),
\[ < X(N^{-2}|x|^2), XN > = -2 N^{-3} |x|^2 < XN, XN > + 2 N^{-2} < \xi_1, XN > \]
\[ = -2 N^{-5} |x|^4 + 2 N^{-2} < \xi_1, XN > . \]
Finally, we obtain
\[ XN = \frac{1}{4} N^{-3} X(\psi^2 + 16|y|^2) = N^{-3} \left\{ |x|^2 \xi_1 + 4X|y|^2 \right\} , \]
which gives
\[ < \xi_1, XN > = N^{-3} \left\{ |x|^4 + 4 < \xi_1, X|y|^2 > \right\} = N^{-3} |x|^4 , \]
where in the last equality we have used (3.3) in Lemma 3.1. Replacing the latter equation in (3.12) we reach the conclusion.

\[ \Box \]

4. Proof of Theorem 1.1

The main goal of this section is proving Theorem 1.1. We begin with some preliminary considerations which have an independent interest. In a Carnot group \( G \) consider a connected bounded open set \( \Omega \subset G \), and the horizontal Laplacian (1.4). In his pioneering paper [Bo] Bony proved the existence of a Green function for \( \mathcal{L} \) and \( \Omega \), i.e., a positive distribution \( G(g,g') = G(g',g) \) such that \( G(g,\cdot) \in C^\infty(\Omega \setminus \{g\}) \) and for which
\[
\begin{align*}
\mathcal{L}G(g,\cdot) &= -\delta_g \quad \text{in} \quad \Omega , \\
G(g,\cdot) &= 0 \quad \text{on} \quad \partial \Omega .
\end{align*}
\]
If we consider the solution to the Dirichlet problem
\[
\begin{align*}
\mathcal{L}u &= -F \quad \text{in} \quad \Omega , \\
u &= 0 \quad \text{on} \quad \partial \Omega ,
\end{align*}
\]
then the following representation formula holds
\[ u(g) = \int_\Omega G(g,g') F(g') \, dg' . \]
Let \( \Gamma(g,g') \) denote the positive fundamental solution of the horizontal Laplacian. Bony's maximum principle implies
\[ G(g,g') \leq \Gamma(g,g') , \quad g,g' \in \Omega , g \neq g'. \]
Using this information in (4.2), and setting \( F = 0 \) in \( \Omega^c \), we conclude
\[ |u(g)| \leq \int_\Omega \Gamma(g,g') |F(g')| \, dg' \leq \int_{\mathcal{H}(g,d)} \Gamma(g,g') |F(g')| \, dg' , \]
where \( B(g, d) \) is a gauge ball with radius \( d = \text{diam}(\Omega) \). Holder’s inequality implies
\[
|u(g)| \leq \left( \int_{B(g, d)} \Gamma(g, g')^{p'} \, dg' \right)^{1/p'} \|F\|_{L^p(\Omega)},
\]
where for \( p > 1 \) we have let \( p' = p/(p - 1) \). Since \( \Gamma(g, g') \) is homogeneous of degree \((2 - Q)\) with respect to the non-isotropic group dilations \([F2]\), we obtain by left-translation and rescaling
\[
\left\{ \int_{B(g, d)} \Gamma(g, g')^{p'} \, dg' \right\}^{1/p'} \leq C(G, p) \left\{ \int_{B(e, d)} N(g)^{p'(2-Q)} \, dg' \right\}^{1/p'}
\]
The latter integral is finite if and only if \( p' < Q/(Q-2) \), which is equivalent to \( p > Q/2 \). If this is the case a simple argument shows that
\[
\left\{ \int_{B(e, d)} N(g)^{p'(2-Q)} \, dg' \right\}^{1/p'} = C(G, p) \frac{d^{2p-Q}}{p}.
\]

These considerations allow to conclude that when \( F \in L^p(\Omega) \) for some \( p > Q/2 \), then the unique solution \( u \) to (4.1) belongs to \( L^\infty(\Omega) \), and one has
\[
(4.3) \quad \sup_{\Omega} |u| \leq C(G, p) \frac{d^{2-Q}}{p} \|F\|_{L^p(\Omega)}.
\]

If in (4.1) instead of the horizontal Laplacian we consider an operator of the type
\[
Lu = \sum_{i,j=1}^m a_{ij} u_{i,j},
\]
with a (real) \( m \times m \) symmetric matrix \((a_{ij})\) satisfying (1.1), then the validity of an estimate similar to (4.3) constitutes a basic open question. For the validity of (4.3) it would suffice that \( L \) and \( \Omega \) admit a Green function satisfying a global estimate of the type
\[
G(g, g') \leq C(G, \Omega) N(g, g')^{2-Q}.
\]

We will come back to these general questions in a forthcoming study.

**Proof of Theorem 1.1.** Let \( G \) be a group of Heisenberg type. We are going to construct a \( m \times m \) matrix-valued function \( g \to (a_{ij}(g)) \) in the form
\[
a_{ij} = \delta_{ij} + \gamma \frac{X_i N X_j N}{|X N|^2}, \quad i, j = 1, ..., m,
\]
where \( N \) is the gauge in (2.3), and the number \( \gamma > 0 \) will be chosen subsequently. We note that \( A(g) = (a_{ij}(g)) \) is symmetric. Moreover, its entries belong to \( L^\infty(G) \). To see this we observe that \( a_{ij}(g) \) are defined and continuous for \( g \in G \setminus E \), where \( E = \{g \in G \mid |X N(g)| = 0\} \). On the other hand Proposition 4.7 in [DGN1] shows that in a group of Heisenberg type the horizontal gradient of any function of the type \( \phi(g) = \psi(|x(g)|^4 + |y(g)|^2) \) can only vanish in a subset of the second layer \( V_2 \). Applying this result to \( \phi = N \) we conclude that
\[
E \subset \{g \in G \mid \xi_1(g) = 0\}.
\]
and therefore the $dg$ measure of $E$ is zero. This proves that $a_{ij} \in L^\infty(G)$. We observe next that if $\gamma > 0$ the matrix $(a_{ij}(g))$ satisfies (1.1). In fact, for any $\zeta \in V_1$ and $dg$-a.e. $g \in G$ we have

$$|\zeta|^2 \leq <A(g)\zeta, \zeta> = |\zeta|^2 + \gamma \frac{<XN(g), \zeta>^2}{|XN|^2} \leq (1 + \gamma) |\zeta|^2.$$ 

In the domain $\Omega = B(e, 1)$ we now try to construct a solution to $Lu = 0$ in the form $u = F(N)$. From (4.4) we obtain

$$Lu = Lu + \gamma \sum_{i,j=1}^{m} u_{ij} \frac{X_i N X_j N}{|XN|^2},$$

where $Lu$ is given by (1.4). The chain rule gives

$$u_{ij} = F''(N) X_i N X_j N + F''(N) N_{ij},$$

and therefore

$$Lu = F''(N) |XN|^2 + F'(N) LN.$$ 

From the proof of (3.4) in [K], see also [CDG], one can assume

$$LN = Q \frac{1}{N} |XN|^2.$$ 

Replacing (4.8) in (4.7) we find the following beautiful formula

$$Lu = |XN|^2 \left\{ F''(N) + \frac{Q - 1}{N} F'(N) \right\}.$$ 

At this point we substitute (4.9), (4.6) into (4.5) obtaining

$$Lu = |XN|^2 \left\{ F''(N) + \frac{Q - 1}{N} F'(N) \right\} + \gamma F''(N) |XN|^2$$

$$+ \gamma \frac{F'(N)}{|XN|^2} \sum_{i,j=1}^{m} N_{ij} X_i N X_j N.$$ 

$$= (1 + \gamma) |XN|^2 \left\{ F''(N) + \frac{Q - 1}{1 + \gamma} F'(N) \right\} + \gamma \frac{F'(N)}{|XN|^2} L_\infty N.$$ 

With equation (4.10) we have reached the crucial point of the proof. It is worth pausing at this moment to mention that, despite the appearances, up to this point we have not really used yet the special structure of groups of Heisenberg type. In fact, (4.10) can be established in every Carnot group provided that we use as “gauge” the function

$$N(g) \overset{def}{=} \Gamma(g, e)^{1/(2 - Q)},$$

where $\Gamma$ is the fundamental solution of the horizontal Laplacian. It is easy to check that such $N$ originates a pseudo-distance, not a distance as it is the case for a group of Heisenberg type. This, however, would have no bearing on the questions considered here.

What really matters instead is that in order to make use of (4.10), and eventually construct a solution to $Lu = 0$, one needs to know that $L_\infty N = 0$. Such deep property is unknown for a general Carnot group and for (4.11), and in fact it is most likely false. For a group of Heisenberg
type, however, we can invoke Theorem 3.4 which allows us to reduce the equation (4.10) to the
following form
\begin{equation}
Lu = (1 + \gamma) |X N|^2 \left\{ F''(N) + \frac{Q - 1}{1 + \gamma} \frac{F'(N)}{N} \right\}.
\end{equation}

It is now clear that \(u = F(N)\) will solve \(Lu = 0\) provided that \(F\) solves the ode
\begin{equation}
F''(N) + \frac{Q - 1}{1 + \gamma} \frac{F'(N)}{N} = 0.
\end{equation}
The equation (4.13) admits the solution \(F(N) = N^\lambda\) with
\begin{equation}
\lambda = 1 - \frac{Q - 1}{1 + \gamma}, \quad \text{or} \quad \gamma = -1 + \frac{Q - 1}{1 - \lambda}.
\end{equation}

We assume that \(0 < \lambda < 1\). With this constraint the number \(\gamma\) defined by (4.14) will be \(> 0\)
provided that \(Q > 2 - \lambda\). This is guaranteed since one always has \(Q = m + 2k \geq 4 > 2 - \lambda\).

We can now easily complete the proof of the theorem. Let \(0 < \epsilon \leq Q\). The function \(N^\lambda\) will
belong to the Sobolev space \(L^{2, Q-\epsilon}(\Omega)\) provided that \((2 - \lambda)(Q - \epsilon) < Q\), or equivalently
\begin{equation}
\lambda = \lambda_\epsilon > 1 - \frac{\epsilon}{Q - \epsilon}.
\end{equation}

This can be easily verified by homogeneity considerations (or by direct calculations). It is
clear that the (4.15) still leaves room for choosing \(\lambda < 1\). In conclusion, letting \(u_\epsilon = 1 - N^{\lambda_\epsilon}\) we
have produced for any \(\epsilon \in (0, Q)\) a function \(u_\epsilon \in L^{2, Q-\epsilon}(\Omega)\) which solves the equation \(L^\epsilon u_\epsilon = 0\)
in \(\Omega\) and takes up continuously the boundary datum \(\phi = 0\). Since on the other hand \(v \equiv 0\) is
also a solution to the same equation, we conclude that uniqueness in the non-homogeneous
Dirichlet problem fails in the class \(L^{2, Q-\epsilon}(\Omega)\), for any \(0 < \epsilon < Q\). As a consequence, an estimate
such as the ABP one cannot hold in such class since, if it did, it would imply such uniqueness.

\[\Box\]

In closing we recall that in his paper [Mi] C. Miranda obtained estimates of the solution of
nondivergence form elliptic equations in terms of the norm \(\|Lu\|_{L^p(\Omega)}\), when \(p > n/2\), provided
that the coefficients belong to the Sobolev space \(W^{1,n}(\Omega)\). Subsequently, Chiarenza, Frasca and
Longo considerably improved on Miranda’s results by establishing the solvability of the Dirichlet
problem in \(W^{2,p}\) under the hypothesis that the coefficients belong to the class \(VMO\) of functions
with vanishing mean oscillation (we recall that \(W^{1,n}\) is contained in \(VMO\)). More recently, the
interior estimates in [CFL], but not the global ones, have been obtained in [BB] for subelliptic
operators of the type \(Lu = \sum_{i,j=1}^m a_{ij}X_iX_j\), with smooth vector fields of Hörmander type [H],
under the assumption that the coefficients \(a_{ij}\) belong to the space \(VMO\) with respect to the
Carnot-Carathéodory metric generated by the \(X_i\)’s. The basic question remains open of finding
geometric conditions of the ground domain that would guarantee the existence of global results,
thus obtaining appropriate extensions of the theorems in [Mi] and [CFL]. In this respect we
note that the existence of the operators \(L^\epsilon\) in Theorem 1.1 would not prevent the possibility of
such results since the coefficients (4.4) in our construction fail to belong to the spaces \(L^{1,Q}(\Omega)\)
or \(VMO\).
ON THE BEST POSSIBLE CHARACTER, ETC.

References


[:] The Monge-Ampère equation in Carnot groups, work in preparation.


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