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ON A CONJECTURE BY KALAI *

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ABSTRACT

We show that monomial ideals generated in degree two satisfy a conjecture by Eisenbud, Green and Harris. In particular we give a partial answer to a conjecture of Kalai by proving that h-vectors of flag Cohen-Macaulay simplicial complexes are h-vectors of Cohen-Macaulay balanced simplicial complexes.

1. Introduction

An unpublished conjecture of Gil Kalai, which was also independently phrased by Jürgen Eckhoff in [Ec] and recently verified by Frohmader [Fr], states that for any flag simplicial complex Δ there exists a balanced simplicial complex Γ with the same f-vector. Here a $(d-1)$-dimensional simplicial complex is balanced if you can use d colors to label its vertices so that no face contains two vertices of the same colour. Kalai's conjecture has also a second part which is still open: If Δ happens to be Cohen-Macaulay (CM), then Γ is required to be CM as well.

In this note we show that for any CM flag simplicial complex Δ there exists a CM balanced simplicial complex Γ with the same h-vector; notice that the equality of the h-vectors implies the equality of the f-vectors only if the simplicial complexes involved have the same dimension.

Equivalently we prove that the h-vector of a CM flag simplicial complex satisfies the Kruskal-Katona's conditions (that is the inequalities satisfied by the f-vector of a simplicial complex, see [St, Theorem 2.1]).

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Such a result has been proved in [CV, Theorem 3.3] under the additional assumption that Δ is vertex decomposable. Other recent developments concerning Kalai's conjecture and related topics can be found in [CN, BV]. To this purpose we will show a stronger statement, Theorem 2.1, namely that the *Eisenbud-Green-Harris conjecture* (**EGH**) holds for quadratic monomial ideals.

Let $S = K[x_1, \ldots, x_n]$ be a polynomial ring over a field K. The **EGH** conjecture, in the general form, states that for every homogeneous ideal I of S containing a regular sequence $f_1, \ldots, f_r$ of degrees $d_1 \leq \cdots \leq d_r$ there exists a homogeneous ideal $J \subseteq S$, with the same Hilbert function as I (i.e. $\mathrm{HF}_I = \mathrm{HF}_J$) and containing $x_1^{d_1}, \ldots, x_r^{d_r}$. Furthermore, by the main theorem in the paper [CL] of Clements and Lindstöm and specifically by Corollary 2 of that paper, the ideal J, when it exists, can be chosen to be the sum of the ideal $(x_1^{d_1}, \ldots, x_r^{d_r})$ and a lex-segment ideal of S. The result of Clements and Lindström recovers Kruskal-Katona theorem when $2 = d_1 = \ldots = d_r$. Recently, these results have been substantially improved by Murai and Mermin in [MM, Theorem 8.1], who dealt with a related question of Evans known as Lex-Plus-Powers Conjecture. We refer to [EGH1] and [EGH2] for the original formulation of the **EGH** conjecture. The only large classes for which the **EGH** conjecture is known are: when $f_1, \ldots, f_r$ are Gröbner basis (by a deformation argument), when $d_i > \sum_{j<i}(d_j - 1)$ for all $i = 2, \ldots, r$ ([CM]) and when each f_i factors as product of linear forms ([Ab, Corollary 4.3] for the case $r = n$, and [Ab] together with the argument in the proof of [CM, Proposition 10] for the general case).

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2. The result

Below $\mathrm{ht}I$ stands for the height of an ideal I and HF_M for the Hilbert function of a graded module M.

Theorem 2.1: *Let $I \subseteq S = K[x_1, \ldots, x_n]$ be a monomial ideal generated in degree 2, of height $\mathrm{ht}I = g$. Then, there exists a monomial ideal $J \in S$, such that $(x_1^2, \ldots, x_g^2) \subseteq J$ and*

$$\mathrm{HF}_I = \mathrm{HF}_J.$$

Furthermore J can be chosen with the same projective dimension as I.

Proof. Since the Hilbert function and the projective dimension are invariant with respect to field extension, we can assume without loss of generality that K is infinite. We will prove that I contains a regular sequence of the form $x_1\ell_1, \ldots, x_g\ell_g$, where ℓ_i is a linear form for every $i \in [g] = \{1, \ldots, g\}$. Then we will infer the theorem by a result in [Ab].

As the minimal primes of a monomial ideal are also monomial, after possibly relabeling the indeterminates, we may assume without loss of generality that $(x_1, \ldots, x_g)$ is a minimal prime of I. Thus, we may decompose the degree 2 component of I as

$$I_2 = x_1V_1 \oplus \ldots \oplus x_gV_g,$$

where each V_i is a linear space generated by indeterminates. Our goal is to find g linear forms $\ell_i \in V_i$, such that:

$(*)$ for all $A \subseteq [g]$, the K-vector space $\langle x_i : i \in A\rangle + \langle \ell_i : i \in [g] \setminus A\rangle$ has dimension g.

Clearly, finding a subset A not satisfying $(*)$ produces a prime ideal containing I of height $< g$, contradicting the fact that $x_1\ell_1, \ldots, x_g\ell_g$ is a regular sequence ([Ma, Theorem 17.4]). So, to see that $(*)$ is equivalent to $x_1\ell_1, \ldots, x_g\ell_g$ being a S-regular sequence (from now on we will just write regular sequence for S-regular sequence), consider the following short exact sequence (where $C = (x_1\ell_1, \dots, x_g\ell_g)$):

$$0 \to (K[x_1, \dots, x_n]/(C : \ell_g))(-1) \to K[x_1, \dots, x_n]/C \to K[x_1, \dots, x_n]/(C + (\ell_g)) \to 0,$$

To conclude the proof of the claim, recall that g homogeneous polynomials form a regular sequence if and only if they generate a height g ideal [Ma, Theorem 17.4].

Notice that $K[x_1, \dots, x_n]/(C : \ell_g) = K[x_1, \dots, x_n]/(B + (x_g))$, where B is an ideal containing $(x_1\ell_1, \dots, x_{g-1}\ell_{g-1})$. Denoting by ℓ_i' the image of ℓ_i by going modulo x_g, we have that

$$K[x_1, \dots, x_n]/(x_1\ell_1, \dots, x_{g-1}\ell_{g-1}, x_g) \cong K[x_1, \dots, x_{g-1}, x_{g+1}, \dots, x_n]/(x_1\ell_1', \dots, x_{g-1}\ell_{g-1}').$$

We can therefore apply the induction on the number of variables and infer that the Krull dimension of $K[x_1, \dots, x_n]/(x_1\ell_1, \dots, x_{g-1}\ell_{g-1}, x_g)$, which bounds from above that of $K[x_1, \dots, x_n]/(C : \ell_g)$, is $n - g$.

Similarly, we can use the induction to infer that $K[x_1, \dots, x_n]/(C + (\ell_g))$ has Krull dimension $n - g$ (notice that, because the assumption that $\langle x_1, \dots, x_{g-1}, \ell_g\rangle$ has dimension g, the image of x_i modulo ℓ_g can be still thought as x_i if $i < g$).

So, both the extremes of the above exact sequence are graded modules of Krull dimension not exceeding $n-g$, or equivalently, the degrees of the corresponding Hilbert polynomials are at most $n-g-1$. Due to the additivity of the Hilbert function over graded exact sequences, the Hilbert polynomial of $K[x_1, \dots, x_n]/C$ has to have degree $< n - g$ (thus Krull dimension $\leq n - g$). Therefore $\mathrm{ht}(C) \geq g$. However we know by Krull's Hauptidealsatz that C may have height at most g, so $\mathrm{ht}(C) = g$ and $x_1\ell_1, \dots, x_g\ell_g$ is a regular sequence.

So we have to seek $\ell_i \in V_i$ satisfying $(*)$. Let $A = \{i_1, \dots, i_a\}$ be a subset of $[g]$. We define $U_A \subseteq \prod_{i\in A} V_i$ to be the following set:

$$U_A = \{(v_{i_1}, \dots, v_{i_a}) \in \prod_{i\in A} V_i \ : \ \langle v_{i_1}, \dots, v_{i_a}\rangle + \langle x_j : j \in [g] \setminus A\rangle \text{ has dimension } g\}.$$

As the condition of linear dependence is obtained by imposing certain determinantal relations to be zero, U_A is a Zariski open set of $\prod_{i\in A} V_i$. Thus the $\widetilde{U}_A$ below is a Zariski open set of $\prod_{i=1}^g V_i$

$$\widetilde{U}_A = U_A \times \prod_{i\in[g]\setminus A} V_i \subseteq \prod_{i=1}^{g} V_i.$$

This construction can be done for every $A \subseteq [g]$, and thus we can define the open set

$$U = \bigcap_{A\subseteq[g]} \widetilde{U}_A \subset \prod_{i=1}^{g} V_i.$$

Any element $(\ell_1, \dots, \ell_g) \in U$ will automatically satisfy $(*)$, so our goal is to show that $U \neq \emptyset$. As $\prod_{i=1}^g V_i$ is irreducible, it is enough to show that all the open sets $\widetilde{U}_A$'s are nonempty. For any $A \subseteq [g]$ we have

$$\dim_K\Big(\sum_{i\in A} V_i + \sum_{j\in[g]\setminus A} \langle x_j\rangle\Big) \geq g, \tag{1}$$

otherwise $(\sum_{i\in A} V_i + \sum_{j\in[g]\setminus A}\langle x_j\rangle)$ would be a prime ideal containing I of height $< g$.

Given $A \subseteq [g]$, we define a bipartite graph G_A. The vertex set of G_A has the following partition: $V(G_A) = \{x_1, \dots, x_n\} \cup \{1, \dots, g\}$, and the edge set of G_A is given by:

$$\{x_i, j\} \in E(G_A) \iff \begin{cases} x_i \in V_j & , \text{ if } j \in A \\ i = j & , \text{ if } j \notin A \end{cases}$$

We fix A and prove that G_A satisfies the hypothesis of the Marriage Theorem. For a subset $B \subseteq V(G_A)$, we denote by $N(B)$ the set of vertices adjacent to some vertex in B. Choose now $B \subseteq \{1,\ldots,g\}$. By applying (1) to the set $A \cap B \subseteq [g]$, we can deduce that

$$\dim_K\Big(\sum_{i\in A\cap B} V_i + \sum_{j\in([g]\setminus A)\cap B}\langle x_j\rangle\Big) \geq \dim_K\Big(\sum_{i\in A\cap B} V_i + \sum_{j\in[g]\setminus(A\cap B)}\langle x_j\rangle\Big) - \dim_K\Big(\sum_{j\in[g]\setminus B}\langle x_j\rangle\Big) \geq |B|.$$

Furthermore, notice that the dimension of the leftmost vector space above is $|N(B)|$, thus we can apply the Marriage Theorem and infer the existence of a matching in G_A of the form $\{x_{i_j}, j\}_{j\in[g]}$. Therefore U_A is nonempty for A nonempty as it contains $(x_{i_j} : j \in A)$, and thus $\widetilde{U}_A$ is nonempty for every A. So we found a regular sequence of quadrics $f_1,\ldots,f_g$ in I consisting of products of linear forms.

Let $\mathrm{pd}(I)$ be the projective dimension of I and assume that $\mathrm{pd}(I) = p-1$. By applying a linear change of coordinates, we may assume that $x_{p+1},\ldots,x_n$ is a S/I-regular sequence. Going modulo $(x_{p+1},\ldots,x_n)$, the image $I' \subseteq K[x_1,\ldots,x_p]$ of I may not be monomial, but still contains a regular sequence of quadrics which are products of linear forms, namely the images of $f_1,\ldots,f_g$. So we find $J' \subseteq K[x_1,\ldots,x_p]$ containing $(x_1^2,\ldots,x_g^2)$ with the same Hilbert function of I' by [Ab, Corollary 4.3] and, when g is less then p, by the same argument used to prove [CM, Proposition 10].

Clearly $\mathrm{pd}(J')= \mathrm{pd}(K[x_1,\ldots,x_p]/J') - 1 \leq p-1$. Furthermore we have $\mathrm{pd}(J') \geq \mathrm{pd}(I) = p-1$ by [CS, Theorem 4.4]. Defining $J = J'K[x_1,\ldots,x_n]$, so we have $(x_1^2,\ldots,x_g^2) \subseteq J$, $\mathrm{pd}(J) = \mathrm{pd}(I)$ and $\mathrm{HF}_I = \mathrm{HF}_J$. ∎

The following example shows that the above proof cannot be extended to prove EGH for all monomial ideals.

Example 2.2: The ideal $I = (x_1^2x_2, x_2^2x_3, x_1x_3^2) \subseteq K[x_1,x_2,x_3]$ does not contain a regular sequence of the form $\ell_1\ell_2\ell_3, q_1q_2q_3$, where all ℓ_i and q_j are linear forms. Elementary direct computations allow one to see that the generators are the only products of three linear forms, which are contained in I. Clearly any choice of two of them does not produce a regular sequence.

The following corollary is the main motivation for this note.

Corollary 2.3: *For any CM flag simplicial complex Δ there exists a CM balanced simplicial complex Γ with the same h-vector.*

Proof. Let g be the height of the Stanley-Reisner ideal I_Δ. By Theorem 2.1, there exists an ideal $J \subseteq S$, containing $(x_1^2,\ldots,x_g^2)$ and with the same Hilbert function and projective dimension as I_Δ. By Auslander-Buchsbaum [Ma, Theorem 19.1], S/J is CM as well, thus all the associated primes of J have the same height by [Ma, Theorem 17.2]. As $(x_1^2,\ldots,x_g^2) \subseteq J$, every associated prime must also contain $(x_1,\ldots,x_g)$, thus the generators of J are monomials in the first g variables. So J is the extension to S of a monomial ideal $J' \subseteq K[x_1,\ldots,x_g]$, whose Hilbert function $\mathrm{HF}_{K[x_1,\ldots,x_g]/J'}$ equals the h-vector of Δ. The CM balanced Γ is the simplicial complex associated to the polarization of J'. ∎

References

[Ab] A. Abedelfatah, *On the Eisenbud-Green-Harris conjecture*, arXiv:1212.2653v1 [math.AC].

[BV] J. Biermann, A. Van Tuyl, *Balanced vertex decomposable simplicial complexes and their h-vectors*, arXiv:1202.0044, 2012.

[CM] G. Caviglia, D. Maclagan, *Some cases of the Eisenbud-Green-Harris conjecture*, Math. Res. Lett. 15 (2008), no. 3, 427-433.

[CS] G. Caviglia, E. Sbarra, *The lex-plus-power inequality for local cohomology modules*, arXiv:1208.2187, 2012.

[CL] G. F. Clements, B. Lindström, *A generalization of a combinatorial theorem of Macaulay*, J. Combinatorial Theory 7, 230-238, 1969.

[CN] D. Cook II, U. Nagel, *Cohen-Macaulay Graphs and Face Vectors of Flag Complexes*, SIAM J. Discrete Math. 26, 89-101, 2012.

[CV] A. Constantinescu, M. Varbaro, *On the h-vectors of Cohen-Macaulay Flag Complexes*, Math. Scand. 112, 86-111, 2013.

[Ec] J. Eckhoff, *Intersection properties of boxes. I. An upper-bound theorem*, Israel J. Math. 62, no. 3, 283-301, 1988.

[EGH1] D. Eisenbud, M. Green and J. Harris, *Higher Castelnuovo Theory*, Asterisque 218, 187-202, 1993.

[EGH2] D. Eisenbud, M. Green and J. Harris, *Cayley-Bacharach theorems and conjectures*, Bull. Amer. Math. Soc. (N.S.) 33, no. 3, 295-324, 1996.

[Fr] A. Frohmader, *Face Vectors of Flag Complexes*, Israel J. Math. 164, 153-164, 2008.

[Ma] H. Matsumura, *Commutative ring theory*, Cambridge studies in advanced mathematics, 1980.

[MM] J. Mermin, S. Murai, *The lex-plus power conjecture holds for pure powers*, Adv. Math. 226, no. 4, 3511-3539, 2011.

[St] R. P. Stanley, *Combinatorics and Commutative Algebra*, Birkhäuser, 1996.