ON HILBERT FUNCTIONS OF GENERAL INTERSECTIONS OF IDEALS

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Abstract. Let I and J be homogeneous ideals in a standard graded polynomial ring. We study upper bounds of the Hilbert function of the intersection of I and $g(J)$, where g is a general change of coordinates. Our main result gives a generalization of Green's hyperplane section theorem.

1. Introduction

Hilbert functions of graded K-algebras are important invariants studied in several areas of mathematics. In the theory of Hilbert functions, one of the most useful tools is Green's hyperplane section theorem, which gives a sharp upper bound for the Hilbert function of R/hR, where R is a standard graded K-algebra and h is a general linear form, in terms of the Hilbert function of R. This result of Green has been extended to the case of general homogeneous polynomials by Herzog and Popescu [HP] and Gasharov [Ga]. In this paper, we study a further generalization of these theorems.

Let K be an infinite field and $S = K[x_1, \dots, x_n]$ a standard graded polynomial ring. Recall that the *Hilbert function* $H(M, -) : \mathbb{Z} \to \mathbb{Z}$ of a finitely generated graded S-module M is the numerical function defined by

$$H(M, d) = \dim_K M_d,$$

where M_d is the graded component of M of degree d. A set W of monomials of S is said to be *lex* if, for all monomials $u, v \in S$ of the same degree, $u \in W$ and $v >_{\mathrm{lex}} u$ imply $v \in W$, where $>_{\mathrm{lex}}$ is the lexicographic order induced by the ordering $x_1 > \cdots > x_n$. We say that a monomial ideal $I \subset S$ is a *lex ideal* if the set of monomials in I is lex. The classical Macaulay's theorem [Mac] guarantees that, for any homogeneous ideal $I \subset S$, there exists a unique lex ideal, denoted by I^{lex}, with the same Hilbert function as I. Green's hyperplane section theorem [Gr] states

Theorem 1.1 (Green's hyperplane section theorem)**.** *Let $I \subset S$ be a homogeneous ideal. For a general linear form $h \in S_1$,*

$$H(I \cap (h), d) \le H(I^{\mathrm{lex}} \cap (x_n), d) \quad \textit{for all } d \ge 0.$$

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Green's hyperplane section theorem is known to be useful to prove several important results on Hilbert functions such as Macaulay's theorem [Mac] and Gotzmann's persistence theorem [Go], see [Gr]. Herzog and Popescu [HP] (in characteristic 0) and Gasharov [Ga] (in positive characteristic) generalized Green's hyperplane section theorem in the following form.

Theorem 1.2 (Herzog–Popescu, Gasharov)**.** *Let $I \subset S$ be a homogeneous ideal. For a general homogeneous polynomial $h \in S$ of degree a,*

$$H(I \cap (h), d) \leq H(I^{\mathrm{lex}} \cap (x_n^a), d) \quad \textit{for all } d \geq 0.$$

We study a generalization of Theorems 1.1 and 1.2. Let $>_{\mathrm{oplex}}$ be the lexicographic order on S induced by the ordering $x_n > \cdots > x_1$. A set W of monomials of S is said to be *opposite lex* if, for all monomials $u, v \in S$ of the same degree, $u \in W$ and $v >_{\mathrm{oplex}} u$ imply $v \in W$. Also, we say that a monomial ideal $I \subset S$ is an *opposite lex ideal* if the set of monomials in I is opposite lex. For a homogeneous ideal $I \subset S$, let I^{oplex} be the opposite lex ideal with the same Hilbert function as I and let $\mathrm{Gin}_\sigma(I)$ be the generic initial ideal ([Ei, §15.9]) of I with respect to a term order $>_\sigma$.

In Section 3 we will prove the following

Theorem 1.3. *Suppose* $\mathrm{char}(K) = 0$*. Let $I \subset S$ and $J \subset S$ be homogeneous ideals such that* $\mathrm{Gin}_{\mathrm{lex}}(J)$ *is lex. For a general change of coordinates g of S,*

$$H(I \cap g(J), d) \leq H(I^{\mathrm{lex}} \cap J^{\mathrm{oplex}}, d) \quad \textit{for all } d \geq 0.$$

Theorems 1.1 and 1.2, assuming that the characteristic is zero, are special cases of the above theorem when J is principal. Note that Theorem 1.3 is sharp since the equality holds if I is lex and J is oplex (Remark 3.5). Note also that if $\mathrm{Gin}_\sigma(I)$ is lex for some term order $>_\sigma$ then $\mathrm{Gin}_{\mathrm{lex}}(J)$ must be lex as well ([Co1, Corollary 1.6]).

Unfortunately, the assumption on J, as well as the assumption on the characteristic of K, in Theorem 1.3 are essential (see Remark 3.6). However, we prove the following result for the product of ideals.

Theorem 1.4. *Suppose* $\mathrm{char}(K) = 0$*. Let $I \subset S$ and $J \subset S$ be homogeneous ideals. For a general change of coordinates g of S,*

$$H(Ig(J), d) \geq H(I^{\mathrm{lex}} J^{\mathrm{oplex}}, d) \quad \textit{for all } d \geq 0.$$

Inspired by Theorems 1.3 and 1.4, we suggest the following conjecture.

Conjecture 1.5. Suppose $\mathrm{char}(K) = 0$. Let $I \subset S$ and $J \subset S$ be homogeneous ideals such that $\mathrm{Gin}_{\mathrm{lex}}(J)$ is lex. For a general change of coordinates g of S,

$$\dim_K \mathrm{Tor}_i(S/I, S/g(J))_d \leq \dim_K \mathrm{Tor}_i(S/I^{\mathrm{lex}}, S/J^{\mathrm{oplex}})_d \quad \text{for all } d \geq 0.$$

Theorems 1.3 and 1.4 show that the conjecture is true if $i = 0$ or $i = 1$. The conjecture is also known to be true when J is generated by linear forms by the result of Conca [Co2, Theorem 4.2]. Theorem 2.8, which we prove later, also provides some evidence supporting the above inequality.

2. Dimension of Tor and general change of coordinates

Let $GL_n(K)$ be the general linear group of invertible $n \times n$ matrices over K. Throughout the paper, we identify each element $h = (a_{ij}) \in GL_n(K)$ with the change of coordinates defined by $h(x_i) = \sum_{j=1}^n a_{ji} x_j$ for all i.

We say that a property (P) holds for a general $g \in GL_n(K)$ if there is a non-empty Zariski open subset $U \subset GL_n(K)$ such that (P) holds for all $g \in U$.

We first prove that, for two homogeneous ideals $I \subset S$ and $J \subset S$, the Hilbert function of $I \cap g(J)$ and that of $Ig(J)$ are well defined for a general $g \in GL_n(K)$, i.e. there exists a non-empty Zariski open subset of $GL_n(K)$ on which the Hilbert function of $I \cap g(J)$ and that of $Ig(J)$ are constant.

Lemma 2.1. *Let $I \subset S$ and $J \subset S$ be homogeneous ideals. For a general change of coordinates $g \in GL_n(K)$, the function $H(I \cap g(J), -)$ and $H(Ig(J), -)$ are well defined.*

Proof. We prove the statement for $I \cap g(J)$ (the proof for $Ig(J)$ is similar). It is enough to prove the same statement for $I + g(J)$. We prove that $\mathrm{in}_{\mathrm{lex}}(I + g(J))$ is constant for a general $g \in GL_n(K)$.

Let t_{kl}, where $1 \le k, l \le n$, be indeterminates, $\tilde{K} = K(t_{kl} : 1 \le k, l \le n)$ the field of fractions of $K[t_{kl} : 1 \le k, l \le n]$ and $A = \tilde{K}[x_1, \dots, x_n]$. Let $\rho : S \to A$ be the ring map induced by $\rho(x_k) = \sum_{l=1}^n t_{lk} x_l$ for $k = 1, 2, \dots, n$, and $\tilde{L} = IA + \rho(J)A \subset A$. Let $L \subset S$ be the monomial ideal with the same monomial generators as $\mathrm{in}_{\mathrm{lex}}(\tilde{L})$. We prove $\mathrm{in}_{\mathrm{lex}}(I + g(J)) = L$ for a general $g \in GL_n(K)$.

Let $f_1, \dots, f_s$ be generators of I and $g_1, \dots, g_t$ those of J. Then the polynomials $f_1, \dots, f_s, \rho(g_1), \dots, \rho(g_t)$ are generators of $\tilde{L}$. By the Buchberger algorithm, one can compute a Gröbner basis of $\tilde{L}$ from $f_1, \dots, f_s, \rho(g_1), \dots, \rho(g_t)$ by finite steps. Consider all elements $h_1, \dots, h_m \in K(t_{kl} : 1 \le k, l \le n)$ which are the coefficient of polynomials (including numerators and denominators of rational functions) that appear in the process of computing a Gröbner basis of $\tilde{L}$ by the Buchberger algorithm. Consider a non-empty Zariski open subset $U \subset GL_n(K)$ such that $h_i(g) \in K \setminus \{0\}$ for any $g \in U$, where $h_i(g)$ is an element obtained from h_i by substituting t_{kl} with entries of g. By construction $\mathrm{in}_{\mathrm{lex}}(I + g(J)) = L$ for every $g \in U$. $\square$

Remark 2.2. The method used to prove the above lemma can be easily generalized to a number of situations. For instance for a general $g \in GL_n(K)$ and a finitely generated graded S-module M, the Hilbert function of $\mathrm{Tor}_i(M, S/g(J))$ is well defined for every i. Let $\mathbb{F} : 0 \xrightarrow{\varphi_{p+1}} \mathbb{F}_p \xrightarrow{\varphi_p} \cdots \longrightarrow \mathbb{F}_1 \xrightarrow{\varphi_1} \mathbb{F}_0 \xrightarrow{\varphi_0} 0$ be a graded free resolution of M. Given a change of coordinates g, one first notes that for every $i = 0, \dots, p$, the Hilbert function $H(\mathrm{Tor}_i(M, S/g(J)), -)$ is equal to the difference between the Hilbert function of $\mathrm{Ker}(\pi_{i-1} \circ \varphi_i)$ and the one of $\varphi_{i+1}(F_{i+1}) + F_i \otimes_S g(J)$

where $\pi_{i-1} : F_{i-1} \to F_{i-1} \otimes_S S/g(J)$ is the canonical projection. Hence we have

$$
\begin{aligned}
&H(\mathrm{Tor}_i(M, S/g(J)), -) = \\
&\qquad H(F_i, -) - H(\varphi_i(F_i) + g(J)F_{i-1}, -) + H(g(J)F_{i-1}, -) \\
&\qquad - H(\varphi_{i+1}(F_{i+1}) + g(J)F_i, -).
\end{aligned} \tag{1}
$$

Clearly $H(F_i, -)$ and $H(g(J)F_{i-1}, -)$ do not depend on g. Thus it is enough to show that, for a general g, the Hilbert functions of $\varphi_i(F_i) + g(J)F_{i-1}$ are well defined for all $i = 0, \dots, p+1$. This can be seen as in Lemma 2.1.

Next, we present two lemmas which will allow us to reduce the proofs of the theorems in the third section to combinatorial considerations regarding Borel-fixed ideals.

The first Lemma is probably clearly true to some experts, but we include its proof for the sake of the exposition. The ideas used in Lemma 2.4 are similar to that of [Ca1, Lemma 2.1] and they rely on the construction of a flat family and on the use of the structure theorem for finitely generated modules over principal ideal domains.

Lemma 2.3. *Let M be a finitely generated graded S-module and $J \subset S$ a homogeneous ideal. For a general change of coordinates $g \in GL_n(K)$ we have that $\dim_K \mathrm{Tor}_i(M, S/g(J))_j \le \dim_K \mathrm{Tor}_i(M, S/J)_j$ for all i and for all j.*

Proof. Let $\mathbb{F}$ be a resolution of M, as in Remark 2.2. Let i, $0 \le i \le p+1$ and notice that, by equation (1), it is sufficient to show: $H(\varphi_i(F_i) + g(J)F_{i-1}, -) \ge H(\varphi_i(F_i) + JF_{i-1}, -)$. We fix a degree d and consider the monomial basis of $(F_{i-1})_d$. Given a change of coordinates $h = (a_{kl}) \in GL_n(K)$ we present the vector space $V_d = (\varphi_i(F_i) + h(J)F_{i-1})_d$ with respect to this basis. The dimension of V_d equals the rank of a matrix whose entries are polynomials in the a_{kl}'s with coefficients in K. Such a rank is maximal when the change of coordinates h is general. $\square$

For a vector $\mathbf{w} = (w_1, \dots, w_n) \in \mathbb{Z}^n_{\ge 0}$, let $\mathrm{in}_{\mathbf{w}}(I)$ be the initial ideal of a homogeneous ideal I with respect to the weight order $>_{\mathbf{w}}$ (see [Ei, p. 345]). Let T be a new indeterminate and $R = S[T]$. For $\mathbf{a} = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\ge 0}$, let $x^{\mathbf{a}} = x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}$ and $(\mathbf{a}, \mathbf{w}) = a_1 w_1 + \cdots + a_n w_n$. For a polynomial $f = \sum_{\mathbf{a} \in \mathbb{Z}^n_{\ge 0}} c_{\mathbf{a}} x^{\mathbf{a}}$, where $c_{\mathbf{a}} \in K$, let $b = \max\{(\mathbf{a}, \mathbf{w}) : c_{\mathbf{a}} \ne 0\}$ and

$$\tilde{f} = T^b \left(\sum_{\mathbf{a} \in \mathbb{Z}^n_{\ge 0}} T^{-(\mathbf{a}, \mathbf{w})} c_{\mathbf{a}} x^{\mathbf{a}} \right) \in R.$$

Note that $\tilde{f}$ can be written as $\tilde{f} = \mathrm{in}_{\mathbf{w}}(f) + Tg$ where $g \in R$. For an ideal $I \subset S$, let $\tilde{I} = (\tilde{f} : f \in I) \subset R$. For $\lambda \in K \setminus \{0\}$, let $D_{\lambda, \mathbf{w}}$ be the diagonal change of coordinates defined by $D_{\lambda, \mathbf{w}}(x_i) = \lambda^{-w_i} x_i$. From the definition, we have

$$R/\big(\tilde{I} + (T)\big) \cong S/\mathrm{in}_{\mathbf{w}}(I)$$

and

$$R/\big(\tilde{I} + (T - \lambda)\big) \cong S/D_{\lambda, \mathbf{w}}(I)$$

where $\lambda \in K \setminus \{0\}$. Moreover $(T-\lambda)$ is a non-zero divisor of $R/\tilde{I}$ for any $\lambda \in K$. See [Ei, §15.8].

Lemma 2.4. *Fix an integer j. Let $\mathbf{w} \in \mathbb{Z}^n_{\geq 0}$, M a finitely generated graded S-module and $J \subset S$ a homogeneous ideal. For a general $\lambda \in K$, one has*

$$\dim_K \operatorname{Tor}_i\big(M, S/\operatorname{in}_{\mathbf{w}}(J)\big)_j \geq \dim_K \operatorname{Tor}_i\big(M, S/D_{\lambda,\mathbf{w}}(J)\big)_j \quad \textit{for all } i.$$

Proof. Consider the ideal $\tilde{J} \subset R$ defined as above. Let $\tilde{M} = M \otimes_S R$ and $T_i = \operatorname{Tor}_i^R(\tilde{M}, R/\tilde{J})$. By the structure theorem for modules over a PID (see [La, p. 149]), we have

$$(T_i)_j \cong K[T]^{a_{ij}} \bigoplus A_{ij}$$

as a finitely generated $K[T]$-module, where $a_{ij} \in \mathbb{Z}_{\geq 0}$ and where A_{ij} is the torsion submodule. Moreover A_{ij} is a module of the form

$$A_{ij} \cong \bigoplus_{h=1}^{b_{ij}} K[T]/(P_h^{i,j}),$$

where $P_h^{i,j}$ is a non-zero polynomial in $K[T]$. Set $l_\lambda = T - \lambda$. Consider the exact sequence

$$0 \longrightarrow R/\tilde{J} \xrightarrow{\cdot l_\lambda} R/\tilde{J} \longrightarrow R/\big((l_\lambda)+\tilde{J}\big) \longrightarrow 0. \tag{2}$$

By considering the long exact sequence induced by $\operatorname{Tor}_i^R(\tilde{M}, -)$, we have the following exact sequence

$$0 \longrightarrow T_i/l_\lambda T_i \longrightarrow \operatorname{Tor}_i^R\big(\tilde{M}, R/\big((l_\lambda)+\tilde{J}\big)\big) \longrightarrow K_{i-1} \longrightarrow 0, \tag{3}$$

where K_{i-1} is the kernel of the map $T_{i-1} \xrightarrow{\cdot l_\lambda} T_{i-1}$. Since l_λ is a regular element for R and $\tilde{M}$, the middle term in (3) is isomorphic to

$$\operatorname{Tor}_i^{R/(l_\lambda)}\big(\tilde{M}/l_\lambda\tilde{M}, R/\big((l_\lambda)+\tilde{J}\big)\big) = \begin{cases} \operatorname{Tor}_i^S\big(M, S/\operatorname{in}_{\mathbf{w}}(J)\big), & \text{if } \lambda = 0, \\ \operatorname{Tor}_i^S\big(M, S/D_{\lambda,\mathbf{w}}(J)\big), & \text{if } \lambda \neq 0 \end{cases}$$

(see [Mat, p. 140]). By taking the graded component of degree j in (3), we obtain

$$\begin{aligned} \dim_K \operatorname{Tor}_i^S\big(M, S/\operatorname{in}_{\mathbf{w}}(J)\big)_j &= a_{ij} + \#\{P_h^{ij} : P_h^{i,j}(0) = 0\} \\ &\quad + \#\{P_h^{i-1,j} : P_h^{i-1,j}(0) = 0\}, \end{aligned} \tag{4}$$

where $\#X$ denotes the cardinality of a finite set X, and

$$\dim_K \operatorname{Tor}_i^S\big(M, S/D_{\lambda,\mathbf{w}}(J)\big)_j = a_{ij} \tag{5}$$

for a general $\lambda \in K$. This proves the desired inequality. □

Corollary 2.5. *With the same notation as in Lemma 2.4, for a general $\lambda \in K$,*

$$\dim_K \operatorname{Tor}_i\big(M, \operatorname{in}_{\mathbf{w}}(J)\big)_j \geq \dim_K \operatorname{Tor}_i\big(M, D_{\lambda,\mathbf{w}}(J)\big)_j \ \textit{for all } i.$$

Proof. For any homogeneous ideal $I \subset S$, by considering the long exact sequence induced by $\mathrm{Tor}_i(M, -)$ from the short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0$ we have

$$\mathrm{Tor}_i(M, I) \cong \mathrm{Tor}_{i+1}(M, S/I) \text{ for } i \geq 1$$

and

$$\dim_K \mathrm{Tor}_0(M, I)_j = \dim_K \mathrm{Tor}_1(M, S/I)_j + \dim_K M_j - \dim_K \mathrm{Tor}_0(M, S/I)_j.$$

Thus by Lemma 2.4 it is enough to prove that

$$\begin{aligned} &\dim_K \mathrm{Tor}_1\big(M, S/\mathrm{in}_{\mathbf{w}}(J)\big)_j - \dim_K \mathrm{Tor}_1\big(M, S/D_{\lambda,\mathbf{w}}(J)\big)_j \\ &\geq \dim_K \mathrm{Tor}_0\big(M, S/\mathrm{in}_{\mathbf{w}}(J)\big)_j - \dim_K \mathrm{Tor}_0\big(M, S/D_{\lambda,\mathbf{w}}(J)\big)_j. \end{aligned}$$

This inequality follows from (4) and (5). $\square$

Proposition 2.6. *Fix an integer j. Let $I \subset S$ and $J \subset S$ be homogeneous ideals. Let $\mathbf{w}, \mathbf{w}' \in \mathbb{Z}^n_{\geq 0}$. For a general change of coordinates $g \in GL_n(K)$,*

(i) $\dim_K \mathrm{Tor}_i(S/I, S/g(J))_j \leq \dim_K \mathrm{Tor}_i(S/\mathrm{in}_{\mathbf{w}}(I), S/\mathrm{in}_{\mathbf{w}'}(J))_j$ *for all i.*
(ii) $\dim_K \mathrm{Tor}_i(I, S/g(J))_j \leq \dim_K \mathrm{Tor}_i(\mathrm{in}_{\mathbf{w}}(I), S/\mathrm{in}_{\mathbf{w}'}(J))_j$ *for all i.*

Proof. We prove (ii) (the proof for (i) is similar). By Lemmas 2.3 and 2.4 and Corollary 2.5, we have

$$\begin{aligned} \dim_K \mathrm{Tor}_i\big(\mathrm{in}_{\mathbf{w}}(I), S/\mathrm{in}_{\mathbf{w}'}(J)\big)_j &\geq \dim_K \mathrm{Tor}_i\big(D_{\lambda_1,\mathbf{w}}(I), S/D_{\lambda_2,\mathbf{w}'}(J)\big)_j \\ &= \dim_K \mathrm{Tor}_i\big(I, S/D^{-1}_{\lambda_1,\mathbf{w}}\big(D_{\lambda_2,\mathbf{w}'}(J)\big)\big)_j \\ &\geq \dim_K \mathrm{Tor}_i\big(I, S/g(J)\big)_j, \end{aligned}$$

as desired, where λ_1, λ_2 are general elements in K. $\square$

Remark 2.7. Let $\mathbf{w}' = (1, 1, \dots, 1)$ and note that the composite of two general changes of coordinates is still general. By replacing J by $h(J)$ for a general change of coordinates h, from Proposition 2.6(i) it follows that

$$\dim_K \mathrm{Tor}_i(S/I, S/h(J))_j \leq \dim_K \mathrm{Tor}_i\big(S/\mathrm{in}_{>_\sigma}(I), S/h(J)\big)_j$$

for any term order $>_\sigma$.

The above fact gives, as a special case, an affirmative answer to [Co2, Question 6.1]. This was originally proved in the thesis of the first author [Ca2]. We mention it here because there seem to be no published article which includes the proof of this fact.

Theorem 2.8. *Fix an integer j. Let $I \subset S$ and $J \subset S$ be homogeneous ideals. For a general change of coordinates $g \in GL_n(K)$,*

(i) $\dim_K \mathrm{Tor}_i(S/I, S/g(J))_j \leq \dim_K \mathrm{Tor}_i(S/\mathrm{Gin}_{\mathrm{lex}}(I), S/\mathrm{Gin}_{\mathrm{oplex}}(J))_j$ *for all i.*
(ii) $\dim_K \mathrm{Tor}_i(I, S/g(J))_j \leq \dim_K \mathrm{Tor}_i(\mathrm{Gin}_{\mathrm{lex}}(I), S/\mathrm{Gin}_{\mathrm{oplex}}(J))_j$ *for all i.*

Proof. Without loss of generality, we may assume $\mathrm{in}_{\mathrm{lex}}(I) = \mathrm{Gin}_{\mathrm{lex}}(I)$ and that $\mathrm{in}_{\mathrm{oplex}}(J) = \mathrm{Gin}_{\mathrm{oplex}}(J)$. It follows from [Ei, Propositin 15.16] that there are vectors $\mathbf{w}, \mathbf{w}' \in \mathbb{Z}^n_{\geq 0}$ such that $\mathrm{in}_{\mathbf{w}}(I) = \mathrm{in}_{\mathrm{lex}}(I)$ and $\mathrm{in}_{\mathbf{w}'}(g(J)) = \mathrm{Gin}_{\mathrm{oplex}}(J)$. Then the desired inequality follows from Proposition 2.6. $\square$

Since $\operatorname{Tor}_0(S/I, S/J) \cong S/(I+J)$ and $\operatorname{Tor}_0(I, S/J) \cong I/IJ$, we have the next corollary.

Corollary 2.9. *Let $I \subset S$ and $J \subset S$ be homogeneous ideals. For a general change of coordinates $g \in GL_n(K)$,*

(i) $H(I \cap g(J), d) \leq H(\operatorname{Gin}_{\mathrm{lex}}(I) \cap \operatorname{Gin}_{\mathrm{oplex}}(J), d)$ *for all* $d \geq 0$.
(ii) $H(Ig(J), d) \geq H(\operatorname{Gin}_{\mathrm{lex}}(I)\operatorname{Gin}_{\mathrm{oplex}}(J), d)$ *for all* $d \geq 0$.

We conclude this section with a result regarding the Krull dimension of certain Tor modules. We show how Theorem 2.8 can be used to give a quick proof of Proposition 2.10, which is a special case (for the variety $X = \mathbb{P}^{n-1}$ and the algebraic group SL_n) of the main Theorem of [MS].

Recall that generic initial ideals are *Borel-fixed*, that is they are fixed under the action of the Borel subgroup of $GL_n(K)$ consisting of all the upper triangular invertible matrices. In particular for an ideal I of S and an upper triangular matrix $b \in GL_n(K)$ one has $b(\operatorname{Gin}_{\mathrm{lex}}(I)) = \operatorname{Gin}_{\mathrm{lex}}(I)$. Similarly, if we denote by *op* the change of coordinates of S which sends x_i to x_{n-i} for all $i = 1, \ldots, n$, we have that $b(op(\operatorname{Gin}_{\mathrm{oplex}}(I))) = op(\operatorname{Gin}_{\mathrm{oplex}}(I))$.

We call *opposite Borel-fixed* an ideal J of S such that $op(J)$ is Borel-fixed (see [Ei, §15.9] for more details on the combinatorial properties of Borel-fixed ideals).

It is easy to see that if J is Borel-fixed, then so is $(x_1, \ldots, x_i) + J$ for every $i = 1, \ldots, n$. Furthermore if j is an integer equal to $\min\{i : x_i \notin J\}$ then $J : x_j$ is also Borel-fixed; in this case I has a minimal generator divisible by x_j or $I = (x_1, \ldots, x_{j-1})$. Analogous statements hold for opposite Borel-fixed ideals.

Let I and J be ideals generated by linear forms. If we assume that I is Borel fixed and that J is opposite Borel fixed, then there exist $1 \leq i, j \leq n$ such that $I = (x_1, \ldots, x_i)$ and $J = (x_j, \ldots, x_n)$. An easy computation shows that the Krull dimension of $\operatorname{Tor}_i(S/I, S/J)$ is always zero when $i > 0$.

More generally one has

Proposition 2.10 (Miller–Speyer). *Let I and J be two homogeneous ideals of S. For a general change of coordinates g, the Krull dimension of* $\operatorname{Tor}_i(S/I, S/g(J))$ *is zero for all* $i > 0$.

Proof. When I or J are equal to (0) or to S the result is obvious. Recall that a finitely generated graded module M has Krull dimension zero if and only if $M_d = 0$ for all d sufficiently large. In virtue of Theorem 2.8 it is enough to show that $\operatorname{Tor}_i(S/I, S/J)$ has Krull dimension zero whenever I is Borel-fixed, J opposite Borel-fixed and $i > 0$. By contradiction, let the pair I, J be a maximal counterexample (with respect to point-wise inclusion). By the above discussion, and by applying *op* if necessary, we can assume that I has a minimal generator of degree greater than 1. Let $j = \min\{h : x_h \notin I\}$ and notice that both $(I : x_j)$ and $(I+(x_j))$ strictly contain I. For every $i > 0$ the short exact sequence $0 \to S/(I : x_j) \to S/I \to S/(I+(x_j)) \to 0$ induces the exact sequence

$$\operatorname{Tor}_i(S/(I : x_j), S/J) \to \operatorname{Tor}_i(S/I, S/J) \to \operatorname{Tor}_i(S/(I+(x_j)), S/J).$$

By the maximality of I, J, the first and the last term have Krull dimension zero. Hence the middle term must have dimension zero as well, contradicting our assumption. □

3. General intersections and general products

In this section, we prove Theorems 1.3 and 1.4. We will assume throughout the rest of the paper $\mathrm{char}(K) = 0$.

A monomial ideal $I \subset S$ is said to be 0-*Borel* (or *strongly stable*) if, for every monomial $ux_j \in I$ and for every $1 \leq i < j$ one has $ux_i \in I$. Note that 0-Borel ideals are precisely all the possible Borel-fixed ideals in characteristic 0. In general, the Borel-fixed property depends on the characteristic of the field and we refer the readers to [Ei, §15.9] for the details. A set $W \subset S$ of monomials in S is said to be 0-*Borel* if the ideal they generate is 0-Borel, or equivalently if for every monomial $ux_j \in W$ and for every $1 \leq i < j$ one has $ux_i \in W$. Similarly we say that a monomial ideal $J \subset S$ is *opposite* 0-*Borel* if for every monomial $ux_j \in J$ and for every $j < i \leq n$ one has $ux_i \in J$.

Let $>_{\mathrm{rev}}$ be the reverse lexicographic order induced by the ordering $x_1 > \cdots > x_n$. We recall the following result [Mu1, Lemma 3.2].

Lemma 3.1. *Let $V = \{v_1, \dots, v_s\} \subset S_d$ be a 0-Borel set of monomials and $W = \{w_1, \dots, w_s\} \subset S_d$ the lex set of monomials, where $v_1 \geq_{\mathrm{rev}} \cdots \geq_{\mathrm{rev}} v_s$ and $w_1 \geq_{\mathrm{rev}} \cdots \geq_{\mathrm{rev}} w_s$. Then $v_i \geq_{\mathrm{rev}} w_i$ for all $i = 1, 2, \dots, s$.*

Since generic initial ideals with respect to $>_{\mathrm{lex}}$ are 0-Borel, the next lemma and Corollary 2.9(i) prove Theorem 1.3.

Lemma 3.2. *Let $I \subset S$ be a 0-Borel ideal and $P \subset S$ an opposite lex ideal. Then $\dim_K(I \cap P)_d \leq \dim_K(I^{\mathrm{lex}} \cap P)_d$ for all $d \geq 0$.*

Proof. Fix a degree d. Let V, W and Q be the sets of monomials of degree d in I, I^{lex} and P respectively. It is enough to prove that $\#V \cap Q \leq \#W \cap Q$.

Observe that Q is the set of the smallest $\#Q$ monomials in S_d with respect to $>_{\mathrm{rev}}$. Let $m = \max_{>_{\mathrm{rev}}} Q$. Then by Lemma 3.1

$$\#V \cap Q = \#\{v \in V : v \leq_{\mathrm{rev}} m\} \leq \#\{w \in W : w \leq_{\mathrm{rev}} m\} = \#W \cap Q,$$

as desired. □

Next, we consider products of ideals. For a monomial $u \in S$, let $\max u$ (respectively, $\min u$) be the maximal (respectively, minimal) integer i such that x_i divides u, where we set $\max 1 = 1$ and $\min 1 = n$. For a monomial ideal $I \subset S$, let $I_{(\leq k)}$ be the K-vector space spanned by all monomials $u \in I$ with $\max u \leq k$.

Lemma 3.3. *Let $I \subset S$ be a 0-Borel ideal and $P \subset S$ an opposite 0-Borel ideal. Let $G(P) = \{u_1, \dots, u_s\}$ be the set of the minimal monomial generators of P. As a K-vector space, IP is the direct sum*

$$IP = \bigoplus_{i=1}^{s} (I_{(\leq \min u_i)}) u_i.$$

Proof. It is enough to prove that, for any monomial $w \in IP$, there is the unique expression $w = f(w)g(w)$ with $f(w) \in I$ and $g(w) \in P$ satisfying

(a) $\max f(w) \leq \min g(w)$.
(b) $g(w) \in G(P)$.

Given any expression $w = fg$ such that $f \in I$ and $g \in P$, since I is 0-Borel and P is opposite 0-Borel, if $\max f > \min g$ then we may replace f by $f\frac{x_{\min g}}{x_{\max f}} \in I$ and replace g by $g\frac{x_{\max f}}{x_{\min g}} \in P$. This fact shows that there is an expression satisfying (a) and (b).

Suppose that the expressions $w = f(w)g(w)$ and $w = f'(w)g'(w)$ satisfy conditions (a) and (b). Then, by (a), $g(w)$ divides $g'(w)$ or $g'(w)$ divides $g(w)$. Since $g(w)$ and $g'(w)$ are generators of P, $g(w) = g'(w)$. Hence the expression is unique. $\square$

Lemma 3.4. *Let $I \subset S$ be a 0-Borel ideal and $P \subset S$ an opposite 0-Borel ideal. Then $\dim_K(IP)_d \geq \dim_K(I^{\mathrm{lex}}P)_d$ for all $d \geq 0$.*

Proof. Lemma 3.1 shows that $\dim_K {I_{(\leq k)}}_d \geq \dim_K {I^{\mathrm{lex}}_{(\leq k)}}_d$ for all k and $d \geq 0$. Then the statement follows from Lemma 3.3. $\square$

Finally we prove Theorem 1.4.

Proof of Theorem 1.4. Let $I' = \mathrm{Gin}_{\mathrm{lex}}(I)$ and $J' = \mathrm{Gin}_{\mathrm{oplex}}(J)$. Since I' is 0-Borel and J' is opposite 0-Borel, by Corollary 2.9(ii) and Lemmas 3.4

$$H(Ig(J), d) \geq H(I'J', d) \geq H(I^{\mathrm{lex}}J', d) \geq H(I^{\mathrm{lex}}J^{\mathrm{oplex}}, d)$$

for all $d \geq 0$. $\square$

Remark 3.5. Theorems 1.3 and 1.4 are sharp. Let $I \subset S$ be a Borel-fixed ideal and $J \subset S$ an ideal satisfying that $h(J) = J$ for any lower triangular matrix $h \in GL_n(K)$. For a general $g \in GL_n(K)$, we have the LU decomposition $g = bh$ where $h \in GL_n(K)$ is a lower triangular matrix and $b \in GL_n(K)$ is an upper triangular matrix. Then as K-vector spaces

$$I \cap g(J) \cong b^{-1}(I) \cap h(J) = I \cap J \text{ and } Ig(J) \cong b^{-1}(I)h(J) = IJ.$$

Thus if I is lex and J is opposite lex then $H(I \cap g(J), d) = H(I \cap J, d)$ and $H(Ig(J), d) = H(IJ, d)$ for all $d \geq 0$.

Remark 3.6. The assumption on $\mathrm{Gin}_{\mathrm{lex}}(J)$ in Theorem 1.3 is necessary. Let $I = (x_1^3, x_1^2x_2, x_1x_2^2, x_2^3) \subset K[x_1, x_2, x_3]$ and $J = (x_3^2, x_3^2x_2, x_3x_2^2, x_2^3) \subset K[x_1, x_2, x_3]$. Then the set of monomials of degree 3 in I^{lex} is $\{x_1^3, x_1^2x_2, x_1^2x_3, x_1x_2^2\}$ and that of J^{oplex} is $\{x_3^3, x_3^2x_2, x_3^2x_1, x_3x_2^2\}$. Hence $H(I^{\mathrm{lex}} \cap J^{\mathrm{oplex}}, 3) = 0$. On the other hand, as we see in Remark 3.5, $H(I \cap g(J), 3) = H(I \cap J, 3) = 1$. Similarly, the assumption on the characteristic of K is needed as one can easily see by considering $\mathrm{char}(K) = p > 0$, $I = (x_1^p, x_2^p) \subset K[x_1, x_2]$ and $J = x_2^p$. In this case we have $H(I^{\mathrm{lex}} \cap J^{\mathrm{oplex}}, p) = 0$, while $H(I \cap g(J), p) = H(g^{-1}(I) \cap J, p) = 1$ since I is fixed under any change of coordinates.

Since $\mathrm{Tor}_0(S/I, S/J) \cong S/(I+J)$ and $\mathrm{Tor}_1(S/I, S/J) \cong (I \cap J)/IJ$ for all homogeneous ideals $I \subset S$ and $J \subset S$, Theorems 1.3 and 1.4 show the next statement.

Remark 3.7. Conjecture 1.5 is true if $i = 0$ or $i = 1$.

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