

A HYPERPLANE RESTRICTION THEOREM AND APPLICATIONS TO REDUCTIONS OF IDEALS

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ABSTRACT. Green's general hyperplane restriction theorem gives a sharp upper bound for the Hilbert function of a standard graded algebra modulo a general linear form. We strengthen Green's result by showing that the linear forms that do not satisfy such estimate belong to a finite union of proper linear spaces. As an application we give a method to derive variations of the Eakin-Sathaye theorem on reductions. In particular, we recover and extend results by O'Carroll on the Eakin-Sathaye theorem for complete and joint reductions.

INTRODUCTION

Let $R = \bigoplus_{d \in \mathbb{N}} R_d$ be a standard graded algebra over an infinite field K . A well-known result of Green [Gr1] provides an upper bound for the dimension of the graded component of degree d of R/lR , where l is a linear form, in terms of the dimension of the graded component of degree d of R . Such a bound is satisfied generically, in the sense that it holds for any linear form in a certain non-empty Zariski open set $U \subseteq \mathbb{A}(R_1)$. This estimate, known as *general hyperplane restriction theorem* is one of the most useful results in the study of Hilbert functions of graded algebras. It plays a central role in modern proofs of many classical theorems on Hilbert functions; Macaulay's characterization of all the possible Hilbert functions of standard graded algebras, Gotzmann's persistence theorem and Gotzmann's regularity theorem [Go] are among those (see for instance [Gr1], [BrHe] Section 4.2 and 4.3, and [Gr2] Section 3).

One of the main results of this paper is a strengthening of the general hyperplane restriction theorem. We show, in Theorem 1.19, that the Zariski open set of linear forms satisfying Green's bound contains the complement of a finite union of proper linear subspaces. The technical aspect of this result is discussed in Section 1 where a more general statement, Theorem 1.13, is presented.

The central part of the proof of Theorem 1.13 follows closely Green's original paper [Gr2], with the main difference that we underline the key-properties (see Definition 1.5) needed in order to build up the inductive steps of the argument. Even though these properties are rather technical, the most common situations in which they are satisfied are quite simple, and allows us to obtain, for instance, Theorem 1.19 as a corollary.

The goal of the first half of this paper, as mentioned above, is to provide a method to substitute the genericity condition for the linear form with a weaker assumption. The following example will perhaps give the reader some motivation of why a modification of Green's result is desirable. Assume for instance that the standard graded algebra R in Green's result is a quotient of the following toric algebra:

$$S = K[X_i Y_j | 0 \leq i \leq n_1, 0 \leq j \leq n_2] \cong K[T_1, \dots, T_{n_1 n_2}] / I.$$

For such an algebra it is reasonable to hope that a linear form of R which is the image of the product of a general linear form in the X_i 's and a general linear form in the Y_j 's may satisfy Green's bound. We will show, as a simple consequence of Theorem 1.13, that this is in fact the case, even though such an element is not general. The linear forms of this type belong in fact to a non-trivial Zariski closed set of the affine space of linear forms $\mathbb{A}(R_1)$.

In the second half of the paper we provide an application of Theorem 1.19 to the theory of reductions of ideals in local rings. Let (A, \mathfrak{m}) be a local ring and let $I \subset A$ be an ideal. A *reduction* of I is an ideal $J \subset I$ such that $I^{n+1} = JI^n$ for some non-negative integer n . The notion of reduction, which was introduced by

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Northcott and Rees in [NoRe], has been widely used in many areas, including multiplicity theory and the theory of blow-up rings. We refer the reader to the dedicated chapter in [HuSw].

Green's general hyperplane restriction theorem can be employed (see [Ca1], or [HuSw] section 8.6) to give a short proof of the Eakin-Sathaye theorem (see [EaSa], [Sa] and also [HoTr]) a well known result on reduction of ideals. Precisely, when $|R/\mathfrak{m}| = \infty$, the main theorem of [EaSa] says that for an integer p large enough so that the number of generators of (I^i) is smaller than $\binom{i+p}{p}$, there exists a reduction (h_1, \dots, h_p) of I such that $I^i = (h_1, \dots, h_p)I^{i-1}$.

The result of [EaSa] has been generalized by O'Carroll [O] (see also [BrEp]) to the case of complete and joint reductions in the sense of Rees. It is worth to note that the elements h_1, \dots, h_p , can be chosen to correspond to general linear forms of the fiber cone ring $R = \bigoplus_{i \geq 0} I^i/\mathfrak{m}I^i$. Theorem 1.13, by strengthening Green's general hyperplane theorem, has the direct consequence of allowing for variations of the Eakin-Sathaye theorem. Specifically, the weakening of the hypothesis on the general linear forms allows us to recover and extend O'Carroll results to a broader range of situations (see Section 2).

1. THE HYPERPLANE RESTRICTION THEOREM

Let R be a standard graded algebra over an infinite field K . We can write R as A/I , where $A = K[X_1, \dots, X_n]$ and I is a homogeneous ideal. In the following, when we say that a property (P) is satisfied by r general linear forms of R we mean that there exists a non-empty Zariski open set $U \subseteq \mathbb{A}(R_1)^r$ such that any r -tuple in U consists of r linear forms satisfying (P) .

Given a positive integer d , any other positive integer c can then be uniquely expressed in term of d as $c = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \dots + \binom{k_1}{1}$, where the k_i 's are non-negative and strictly decreasing, i.e. $k_d > k_{d-1} > \dots > k_1 \geq 0$. This way of writing c is called the d 'th *Macaulay representation* of c , and the k_i 's are called the d 'th *Macaulay coefficients* of c . The integer $c_{\langle d \rangle}$ is defined to be $c_{\langle d \rangle} = \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \dots + \binom{k_1-1}{1}$.

Mark Green proved the following:

Theorem (General Hyperplane Restriction Theorem). *Let R be a standard graded algebra over an infinite field K , let d be a degree and let l be a general linear form of R . Then*

$$(1.1) \quad \dim_k(R/lR)_d \leq (\dim_K R)_{\langle d \rangle}.$$

The above result was first proved in [Gr1] with no assumption on the characteristic of the base field K . For the case $\text{char}(K) = 0$, a more combinatorial and perhaps simpler proof can be found in [Gr2]; it uses generic initial ideals with respect to the lexicographic order and it gives at the same time a proof of Macaulay's estimate on Hilbert functions. Despite the extensive literature, I am not aware of any argument based solely on generic initial ideals that would derive (1.1) in any characteristic.

It is important to recall that the numerical bound (1.1) can be also interpreted in the following way: let $A = K[X_1, \dots, X_n]$ and let $I \subset A$ be a homogeneous ideal. Define $J = I^{\text{lex}} \subset A$ to be the unique lexicographic ideal with the same Hilbert function as I . Let c be the dimension, as a K -vector space, of $(A/I)_d$. By definition we also have that $\dim_K(A/J)_d = c$. It is possible to show that $\dim_K(A/(J + (X_n)))_d = c_{\langle d \rangle}$. Then, for any degree d , the general hyperplane restriction theorem is equivalent to the statement that if l is general then

$$(1.2) \quad \dim_K(A/I + (l))_d \leq \dim_K(A/I^{\text{lex}} + (X_n))_d.$$

The inequality (1.2) has been generalized by several authors. Aldo Conca [Co] has proved that when characteristic of K is zero and l_1, \dots, l_r are general linear forms one has

$$(1.3) \quad \dim_K \text{Tor}_i(A/I, A/(l_1, \dots, l_r))_d \leq \dim_K \text{Tor}_i(A/I^{\text{lex}}, A/(X_n, \dots, X_r))_d \quad \text{for all } i \text{ and } d.$$

Conca's result, when $i = 0$, corresponds to the characteristic zero case of Green's theorem. Unfortunately, the method used in [Co] requires the use of general initial ideals and, as I mention above, this puts some restriction on the possible characteristic of K . In a different direction Herzog and Popescu [HP] and Gasharov [Ga] extended (1.2) by showing that the next inequality holds for a general form f of A of degree c

$$(1.4) \quad \dim_K(A/I + (f))_d \leq \dim_K(A/I^{\text{lex}} + (X_n^c))_d.$$

Further generalizations in this direction can be found in [CaMu].

The goal of the remaining part of this section is to show how the assumption of generality on the linear form satisfying (1.1), or equivalently (1.2), can be relaxed. We first introduce some notation.

Let R be a standard graded algebra, we denote by \mathfrak{m} its homogeneous maximal ideal and let $\mathbf{l} = l_1, \dots, l_r$ be a sequence of linear forms. For every sequence $\mathbf{o} = o_1, \dots, o_m$ and $1 \leq m \leq r$ with $o_i \in \{c, s\}$, we construct an ideal $I_{\mathbf{l}, \mathbf{o}}$ by recursively considering colons and sums of the above linear forms. Precisely, when \mathbf{o} equals c or s we set $I_{\mathbf{l}, \mathbf{o}}$ to be $(0) : l_1$ and (l_1) respectively; similarly if $\bar{\mathbf{o}} = o_1, \dots, o_i$ and $\mathbf{o} = \bar{\mathbf{o}}, o_{i+1}$ we set $I_{\mathbf{l}, \mathbf{o}}$ to be $I_{\mathbf{l}, \bar{\mathbf{o}}} : l_{i+1}$ or $I_{\mathbf{l}, \bar{\mathbf{o}}} + l_{i+1}$ depending whether o_{i+1} is c or s . We also let $|\mathbf{o}|_c$ and $|\mathbf{o}|_s$ be, respectively, the number of c and of s in \mathbf{o} . For simplicity, whenever it will be clear from the context what the sequence \mathbf{l} is, we will just write $I_{\mathbf{o}}$ instead of $I_{\mathbf{l}, \mathbf{o}}$.

Definition 1.5. We say that l_1, \dots, l_r satisfy property **(Gr, d)** (i.e. they are suitable for an hyperplane restriction theorem in degree d), if for every sequence $\mathbf{o} = o_1, \dots, o_i$ with $1 \leq i \leq r$, the following hold:

- (i) If $\mathfrak{m} \not\subseteq I_{\mathbf{o}}$ and $i < r$, then $l_{i+1} \notin I_{\mathbf{o}}$.
- (ii) If $i = r$ and $|\mathbf{o}|_c < d$ then $\mathfrak{m} \subseteq I_{\mathbf{o}}$.
- (iii) If $i \leq r - 2$ then $\dim_K(I_{\mathbf{o}, c, s})_{d-|\mathbf{o}|_c-1} \leq \dim_K(I_{\mathbf{o}, s, c})_{d-|\mathbf{o}|_c-1}$.

Remark 1.6. Let $n = \dim_K R_1$. If property (i) holds, then property (ii) is automatically satisfied if $r \geq n + d - 1$; in particular, in this case, l_1, \dots, l_r generates \mathfrak{m} . Property (iii) is implied by the next stronger condition.

- (iv) $\dim_K(I_{\mathbf{o}, c, s})_{d-|\mathbf{o}|_c-1} = \dim_K((I_{\mathbf{o}} : l_{i+2}) + l_{i+1})_{d-|\mathbf{o}|_c-1}$.

To see that this is the case, notice that $(I_{\mathbf{o}} : l_{i+2}) + l_{i+1} \subseteq I_{\mathbf{o}, s, c}$ and thus (iv) gives: $\dim_K(I_{\mathbf{o}, c, s})_{d-|\mathbf{o}|_c-1} = \dim_K((I_{\mathbf{o}} : l_{i+2}) + l_{i+1})_{d-|\mathbf{o}|_c-1} \leq \dim_K(I_{\mathbf{o}, s, c})_{d-|\mathbf{o}|_c-1}$.

At a first sight the properties **(Gr, d)** may not seem easy to verify. However, there are several examples for which it is not hard to find linear forms satisfying them. For instance let $r = n + d - 1$, and assume that the linear forms l_1, \dots, l_r span \mathfrak{m} and that for each \mathbf{o} the Hilbert functions of the ideals $I_{\mathbf{o}}$ at the degrees between 0 and $d - |\mathbf{o}|_c - 1$ do not depend on the order of the l_1, \dots, l_r . This implies immediately (i) and (iv), hence l_1, \dots, l_r satisfy **(Gr, d)**. We summarize the above considerations in the following lemma.

Lemma 1.7. *Let $n = \dim_K R_1$, d be a degree, $r \geq n + d - 1$, and l_1, \dots, l_r linear forms of R generating \mathfrak{m} . If for every sequence $\mathbf{o} = c_1, \dots, c_i$ with $i \leq r$, and for every $j \in \{0, \dots, d - |\mathbf{o}|_c - 1\}$ we have that $\dim_K(I_{\mathbf{o}})_j$ is independent of the order of the l_s 's then l_1, \dots, l_r satisfy **(Gr, d)**.*

We complete our discussion of the properties **(Gr, d)** with the following two propositions, which show a simple case in which the assumptions of Lemma 1.7 are satisfied. The experts will be able to see immediately why the conclusion of Proposition 1.8 and Proposition 1.9 are obvious, and we include for the sake of the exposition, some concise explanations of why is this the case.

Proposition 1.8. *Let $R = K[X_1, \dots, X_n]/J$ be a standard graded algebra over an algebraically closed field K . Let $V \subset \mathbb{A}(R_1)$ be an irreducible variety. Then for r -linear forms of R that are general points of V , and for every sequence $\mathbf{o} = o_1, \dots, o_r$ the Hilbert function of $I_{\mathbf{o}}$ is well defined, equivalently there exists a non-empty Zariski open subset of V^r on which the Hilbert function of $I_{\mathbf{o}}$ is constant.*

Proof. We consider the coordinate ring of V , say $S_V = K[Y_1, \dots, Y_n]/I_V$ and more generally the coordinates ring $S_{V^r} = K[Y_{1,1}, \dots, Y_{1,n}, \dots, Y_{r,1}, \dots, Y_{r,n}]/I_{V^r}$ of V^r . Since V is irreducible, so is V^r , thus the defining ideal I_{V^r} is prime. Let \mathbb{K} be the fraction field of S_{V^r} . For every i, j we denote with $y_{i,j}$ the image in \mathbb{K} of $Y_{i,j}$. Let $\tilde{\mathbf{l}}$ be the sequence of linear forms of $\tilde{l}_1, \dots, \tilde{l}_r$ of $\mathbb{K}[X_1, \dots, X_n]$, where, for $i = 1, \dots, r$, we have set $\tilde{l}_i = y_{i,1}X_1 + \dots + y_{i,n}X_n$. There is a standard way, using Gröbner bases, to explicitly compute a reduced Gröbner basis (with respect to the reverse lexicographic order) of the pre-image in $\mathbb{K}[X_1, \dots, X_n]$ of the ideal $I_{\tilde{\mathbf{l}}, \mathbf{o}} \subseteq R \otimes_K \mathbb{K}$. We consider all the coefficients $\alpha_1, \dots, \alpha_q$ of all the monomials appearing in all the polynomials involved in such computation, we let U_i be the non-empty Zariski open set of V^r where the rational function α_i is non-zero. Then we consider $U = \bigcap_i^q U_i$ which is not empty since V^r is irreducible. \square

Proposition 1.9. *With the same assumption as Proposition 1.8, for r -linear forms of R that are general points of V , and for every sequence $\mathbf{o} = o_1, \dots, o_r$ the Hilbert function of $I_{\mathbf{o}}$ is independent of the order of the linear forms.*

The following examples are a strait forward application of Lemma 1.7.

Example 1.10. Let $R = K[X, Y, Z]/I$ be a standard graded algebra over an algebraically field K . Then for a general choice of $\lambda_1, \dots, \lambda_{d+2}$ the linear forms $l_i = X + \lambda_i Y + \lambda_i^2 Z$ satisfy (\mathbf{Gr}, d) .

Example 1.11. We now give a list of examples of graded algebras and corresponding sets of linear forms satisfying (\mathbf{Gr}, d) . These example will be relevant to the discussion of the next section. They will correspond to variations of the Eakin-Sathaye theorem.

Let R be a standard graded algebra, $\dim_K R_1 = n$, $|K| = \infty$. The following are examples of r linear forms with $r \geq d + n - 1$, satisfying (\mathbf{Gr}, d) . Note that all the homomorphisms mentioned below are assumed to send the monomials generating each sub-algebra to linear forms.

- (A) With no further assumptions on R the r linear forms can be taken to be general.
- (B) Assume R to be the homomorphic image of the Segre ring:

$$S = K[X_{1,i_1} \cdot X_{2,i_2} \cdots X_{s,i_s} | 1 \leq i_1 \leq n_1, \dots, 1 \leq i_s \leq n_s].$$

Then the r linear forms can be taken to be the images of $l_1 \cdots l_s$, where l_i is a general linear form of $K[X_{i,1}, \dots, X_{i,n_i}]$.

- (C) Assume that $\text{char}(K) = 0$ and that R is the homomorphic image of the Veronese ring: $S = K[X_1^{a_1} \cdots X_s^{a_s} | \sum_{i=1}^s a_i = b$ and $a_i \geq 0]$. Then the r linear forms can be taken to be the images of l^b , where l is a general linear form of $K[X_1, \dots, X_s]$.
- (D) Assume that $\text{char}(K) = 0$ and that R is the homomorphic image of Segre products of Veronese rings:

$$S = K \left[\prod_{\substack{1 \leq i \leq s \\ 1 \leq j \leq n_i}} X_{i,j}^{a_{i,j}} \text{ such that } \sum_j a_{i,j} = b_i \text{ and } a_{i,j} \geq 0 \right].$$

Then the r linear forms can be taken to be the images of $l_1^{b_1} \cdots l_s^{b_s}$, where l_i is a general linear form of $K[X_{i,1}, \dots, X_{i,n_i}]$.

- (E) Assume that $\text{char}(K) = 0$ and that R is the homomorphic image of the following toric ring:

$$S = K[X_{i_1} \cdots X_{i_s} | 1 \leq i_1 \leq n_1, \dots, 1 \leq i_s \leq n_s \text{ and } n_1 \leq n_2 \leq \dots \leq n_s].$$

Then the r linear forms can be taken to be the images of $l_1(l_1 + l_2) \cdots (l_1 + l_2 + \dots + l_s)$, where l_i is a general linear form of $K[X_{n_{i-1}}, \dots, X_{n_i}]$.

Remark 1.12. Note that the characteristic assumption in (C),(D) and (E) is essential.

Let $R = K[X_1^2, X_1 X_2, X_2^2]/(X_1^2, X_2^2) \cong K[Y_1, Y_2, Y_3]/(Y_1, Y_3, Y_2^2 - Y_1 Y_3)$ and assume $\text{char}(K) = 2$. This correspond to the case $s = 2$ and $b = 2$ of example (C). The square of a general linear form of $K[X_1, X_2]$ can be written as $X_1^2 + \lambda X_2^2$ and it has a zero image in R . Property (2) of (\mathbf{Gr}, d) is not satisfied. Moreover, a zero linear form clearly does not satisfy Green's estimate.

We can now prove the desired hyperplane restriction theorem. The structure of the proof follows the outline of [Gr2].

Theorem 1.13. *Let R be a standard graded algebra and let l_1, \dots, l_r be linear forms satisfying (\mathbf{Gr}, d) . Then*

$$\dim_K(R/(l_1))_d \leq (\dim_K(R_d))_{(d)}.$$

Proof. We let $\mathbf{I}_{i,j}$ to be the set of all ideals $I_{\mathbf{o}}$ such that $|\mathbf{o}| = i$ and $|\mathbf{o}|_c = j$. In order to prove the theorem it is enough to show:

Claim. For any $I \in \mathbf{I}_{i,j}$ with $i < r$ we have:

$$(1.14) \quad \dim_K(R/(I + (l_{i+1})))_{d-j} \leq (\dim_K(R/I)_{d-j})_{\langle d-j \rangle}.$$

First of all, we show that the claim holds for all the ideals in $\mathbf{I}_{r-1,j}$ and in $\mathbf{I}_{i,d-1}$. By part (ii) of (\mathbf{Gr}, d) , since $I \in \mathbf{I}_{r-1,j}$ then $\mathbf{m} \subseteq (I + (l_r)) \in \mathbf{I}_{r,j}$. Because $j < d$ we have $(R/(I + (l_r)))_{d-j} = 0$ and, therefore, the inequality (1.14) holds. If $I \in \mathbf{I}_{i,d-1}$ the inequality (1.14) becomes $\dim_K(R/(I + (l_{i+1})))_1 \leq \dim_K(R/I)_1 - 1$ and it follows from the part (i) of (\mathbf{Gr}, d) .

We do a decreasing induction on the double index of $\mathbf{I}_{i,j}$.

Let $I \in \mathbf{I}_{a,d-b}$ with $a < r-1$ and $b > 1$. By induction we know that (1.14) holds for $(I + (l_{a+1})) \in \mathbf{I}_{a+1,d-b}$ and for $(I : l_{a+1}) \in \mathbf{I}_{a+1,d-b+1}$. Consider the sequence below:

$$0 \rightarrow \frac{R}{(I + (l_{a+1})) : l_{a+2}}(-1) \xrightarrow{\cdot l_{a+2}} \frac{R}{I + (l_{a+1})} \rightarrow \frac{R}{I + (l_{a+1}) + (l_{a+2})} \rightarrow 0.$$

By looking at the graded component of degree b we get:

$$\begin{aligned} \dim_K \left(\frac{R}{I + (l_{a+1})} \right)_b &= \dim_K \left(\frac{R}{(I + (l_{a+1})) : l_{a+2}} \right)_{b-1} + \\ &+ \dim_K \left(\frac{R}{I + (l_{a+1}) + (l_{a+2})} \right)_b. \end{aligned}$$

Property (iii) of (\mathbf{Gr}, d) implies

$$\dim_K \left(\frac{R}{(I + (l_{a+1})) : l_{a+2}} \right)_{b-1} \leq \dim_K \left(\frac{R}{(I : l_{a+1}) + (l_{a+2})} \right)_{b-1},$$

and by using the inductive assumption on $I : l_{a+1}$ and on $I + (l_{a+1})$ we know that $\dim_K \left(\frac{R}{I + (l_{a+1})} \right)_b \leq \left(\dim_K \left(\frac{R}{I : l_{a+1}} \right)_{b-1} \right)_{\langle b-1 \rangle} + \left(\dim_K \left(\frac{R}{I + (l_{a+1})} \right)_b \right)_{\langle b \rangle}$.

To simplify the notation, set $c = \dim_K(R/I)_b$ and $c_H = \dim_K(R/(I + (l_1)))_b$. From the short exact sequence

$$0 \rightarrow \frac{R}{I : l_{a+1}}(-1) \xrightarrow{\cdot l_{a+1}} \frac{R}{I} \rightarrow \frac{R}{I + (l_{a+1})} \rightarrow 0$$

we know that $\dim_K \left(\frac{R}{I : l_{a+1}} \right)_{b-1} = c - c_H$, therefore the above upper bound for $\dim_K \left(\frac{R}{I + (l_{a+1})} \right)_b$ becomes:

$$(1.15) \quad c_H \leq (c_H)_{\langle b \rangle} + (c - c_H)_{\langle b-1 \rangle}.$$

Write $c_H = \binom{k_b}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_\delta}{\delta}$. The inequality of the claim, i.e. $c_H \leq c_{\langle b \rangle}$, is equivalent to $c \geq \binom{k_b+1}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_\delta+1}{\delta}$.

If the claim fails we have:

$$(1.16) \quad c - c_H < \binom{k_b}{b-1} + \binom{k_{b-1}}{b-2} + \cdots + \binom{k_\delta}{\delta-1}.$$

We use (1.15) to derive a contradiction. There are two cases to consider.

If $\delta = 1$ then (1.16) becomes $c - c_H \leq \binom{k_b}{b-1} + \binom{k_{b-1}}{b-2} + \cdots + \binom{k_2}{1}$.

Thus

$$\begin{aligned} (c - c_H)_{\langle b-1 \rangle} &\leq \binom{k_b-1}{b-1} + \binom{k_{b-1}-1}{b-2} + \cdots + \binom{k_2-1}{1} && \text{and} \\ (c_H)_{\langle b \rangle} &\leq \binom{k_b-1}{b} + \binom{k_{b-1}-1}{b-1} + \cdots + \binom{k_2-1}{2} + \binom{k_1-1}{1}. \end{aligned}$$

By adding these two inequalities, (1.15) gives

$$c_H \leq \binom{k_b}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_1-1}{1} < c_H,$$

which is a contradiction.

If $\delta > 1$ then the equation (1.16) is $c - c_H < \binom{k_b}{b-1} + \binom{k_{b-1}}{b-2} + \cdots + \binom{k_\delta}{\delta-1}$ and since $k_\delta - 1 > \delta - 1$ applying b_{-1} the strict inequality is preserved and gives

$$(c - c_H)_{(b-1)} < \binom{k_b - 1}{b-1} + \binom{k_{b-1} - 1}{b-2} + \cdots + \binom{k_\delta - 1}{\delta-1}.$$

Adding the last inequality with $(c_H)_{(b)} \leq \binom{k_b-1}{b} + \binom{k_{b-1}-1}{b-1} + \cdots + \binom{k_\delta-1}{\delta}$ we obtain the following contradiction

$$c_H < \binom{k_b}{b} + \binom{k_{b-1}}{b-1} + \cdots + \binom{k_1}{1} = c_H.$$

□

A direct consequence of Theorem 1.13 is the corollary below.

Corollary 1.17. *Let R be a standard graded algebra and let l_1, \dots, l_r be linear forms satisfying (\mathbf{Gr}, d) , and let the Macaulay representation of $\dim_K(R_d)$ be $\binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots + \binom{k_1}{1}$. Then for any p such that $1 \leq p \leq r$ we have*

$$\dim_K(R/(l_1, \dots, l_p))_d \leq \binom{k_d - p}{d} + \binom{k_{d-1} - p}{d-1} + \cdots + \binom{k_1 - p}{1}.$$

Proof. By Theorem 1.13 we have $\dim_K(R/(l_1))_d \leq \binom{k_d-1}{d} + \binom{k_{d-1}-1}{d-1} + \cdots + \binom{k_1-1}{1}$. Note that the images of l_2, \dots, l_r satisfy (\mathbf{Gr}, d) for $R/(l_1)$. Thus we apply Theorem 1.13 and obtain the result by induction. □

We combine Corollary 1.17 and Proposition 1.9 in the following result.

Theorem 1.18. *Let $R = K[X_1, \dots, X_n]/I$ be a standard graded algebra over an algebraically closed field K and let $V \subset \mathbb{A}(R_1)$ be an irreducible variety spanning $\mathbb{A}(R_1)$. Then for any p linear forms of R that are general points of V we have:*

$$\dim_K(R/(l_1, \dots, l_p))_d \leq \binom{k_d - p}{d} + \binom{k_{d-1} - p}{d-1} + \cdots + \binom{k_1 - p}{1}.$$

In particular we obtain a description of the open set where Green's estimate holds:

Theorem 1.19 (Hyperplane Restriction). *Let R be a standard graded algebra over an algebraically closed field K , and let $\mathbb{A}(R_1)$ be the affine space of linear forms of R_1 . Then there exist proper linear subspaces V_1, \dots, V_m of $\mathbb{A}(R_1)$ such that for any form $l \notin (\cup V_i)$ we have:*

$$(*) \quad \dim_k(R/lR)_d \leq (\dim_K R_d)_{(d)}.$$

Proof. First of all, note that the linear forms not satisfying the above inequality constitute a Zariski closed set of $\mathbb{A}(R_1)$. By contradiction we can assume that such a closed set has an irreducible component, say V , not contained in any finite union of proper linear subspaces. This implies that V spans $\mathbb{A}(R_1)$, and by using Theorem 1.18, we derive a contradiction. □

2. VARIATIONS OF THE EAKIN-SATHAYE THEOREM

We now prove a general version of the Eakin-Sathaye theorem.

Theorem 2.1. *Let (A, m) be a quasi-local ring with infinite residue field K . Let I be an ideal of A . Let i and p be positive integers. If the number of minimal generators of I^i , denoted by $v(I^i)$, satisfies $v(I^i) < \binom{i+p}{i}$ then*

(a) (Eakin-Sathaye) *There are h_1, \dots, h_p in I such that $I^i = (h_1, \dots, h_p)I^{i-1}$.*

Moreover:

(b) (O'Carroll) *If $I = I_1 \cdots I_s$, where I_j 's are ideals of R , then we can find the elements h_j of the form $l_1 \cdots l_s$ with $l_i \in I_i$.*

- (c) Assume $\text{char}(K) = 0$. If $I = J^b$, where J is an ideal of A , then we can find the elements h_j 's of the form l^b with $l \in I$.
- (d) Assume $\text{char}(K) = 0$. If $I = I_1^{b_1} \cdots I_s^{b_s}$, where I_j 's are ideals of A , then we can find the elements h_j of the form $l_1^{b_1} \cdots l_s^{b_s}$ with $l_i \in I_i$.
- (e) Assume $\text{char}(K) = 0$. If $I = I_1(I_1 + I_2) \cdots (I_1 + \cdots + I_s)$, where I_j 's are ideals of A , then we can find the elements h_j of the form $l_1(l_1 + l_2) \cdots (l_1 + \cdots + l_s)$ with $l_i \in I_i$.

Proof. First of all, note that since $v(I^i)$ is finite, without loss of generality we can assume that I is also finitely generated: in fact, if $H \subseteq I$ is a finitely generated ideal such that $H^i = I^i$ the result for H implies the one for I . Similarly, we can also assume that the ideals I_j of (b),(d) and (e) and the ideal J of (c) are finitely generated. By the use of Nakayama's Lemma, we can replace I by the homogeneous maximal ideal of the fiber cone $R = \bigoplus_{i \geq 0} I^i/mI^i$. Note that R is a standard graded algebra finitely generated over the infinite field $R/m = K$. Moreover, the algebras R of (a),(b),(c),(d),(e) satisfy the properties of the Example 1.11 parts (A),(B),(C),(D), and (E) respectively. Let l_1, \dots, l_r as in Example 1.11 and assume also that $p \leq r$. The theorem is proved if we can show that $(R/(l_1, \dots, l_p))_i = 0$. Note that $\dim_K R_i \leq \binom{i+p}{i} - 1 = \binom{i+p-1}{i} + \binom{i+p-2}{i-1} + \cdots + \binom{i+p-j}{i-j+1} + \cdots + \binom{r}{1}$. This can be proved directly. In fact, one can first order the array of Macaulay coefficients using the lexicographic order and then note that the previous array of $(i+p, 0, \dots, 0)$ is given by $(i+p-1, i+p-2, \dots, p)$. By Corollary 1.17 we deduce

$$\dim_K (R/(l_1, \dots, l_p))_i \leq \binom{i-1}{i} + \binom{i-2}{i-1} + \cdots + \binom{0}{1}.$$

The term on the right hand side is zero and therefore the theorem is proved. \square

REFERENCES

- [BrEp] P. Brennan, N. Epstein. *Noether normalizations, reductions of ideals, and matroids*, Proc. Amer. Math. Soc. **139** (2011), 2671-2680.
- [BrHe] W. Bruns, J. Herzog. *Cohen-Macaulay rings, Cambridge Studies in Mathematics* **39**, Cambridge University Press, Cambridge, 1993.
- [Ca1] G. Caviglia. *A Theorem of Eakin and Sathaye and Green's Hyperplane Restriction Theorem* Proceedings of the Lisbon Conference on Commutative Algebra 2003.
- [Ca2] G. Caviglia. *Koszul algebras, Castelnuovo-Mumford regularity and generic initial ideals*, Thesis, University of Kansas, 2004.
- [CaMu] G. Caviglia, S. Murai. *On Hilbert function of general intersection of ideals*, arXiv:1303.6030 [math.AC].
- [Co] A. Conca. *Koszul homology and extremal properties of Gin and Lex*. Trans. Amer. Math. Soc. **356** (2004), no. 7, 2945-2961.
- [EaSa] P. Eakin, A. Sathaye. *Prestable ideals*. J. Algebra **45** (1976), no. 2, 439-454.
- [Ga] V. Gasharov. *Hilbert functions and homogeneous generic forms, II*. Compositio Math. **116** (1999), 167-172.
- [Go] G. Gotzmann. Eine Bedingung für die Flachheit und das Hilbertpolynom eines graduierten Ringes *Math. Z.* **158** (1978), 61-70.
- [Gr1] M. Green. *Restrictions of linear series to hyperplanes, and some results of Macaulay and Gotzmann*. Algebraic curves and projective geometry (Trento, 1988), 76-86, Lecture Notes in Math., **1389**, Springer, Berlin, 1989.
- [Gr2] M. Green. *Generic initial ideals*. Six lectures on commutative algebra. Papers from the Summer School on Commutative Algebra held in Bellaterra, July 16-26, 1996. Progress in Mathematics, **166**, 119-186.
- [HP] J. Herzog and D. Popescu. *Hilbert functions and generic forms*. Compositio Math, **113** (1998), 1-22.
- [HuSw] C. Huneke, I. Swanson. *Integral closure of ideals, rings, and modules*. London Mathematical Society Lecture Note Series, **336**. Cambridge University Press, Cambridge, 2006.
- [HoTr] Le Tuan Hoa, Ngo Viet Trung. *Borel-fixed ideals and reduction number*. J. Algebra **270** (2003), no. 1, 335-346.
- [NoRe] D. G. Northcott, D. Rees. *Reductions of ideals in local rings*. Proc. Camb. Phil. Soc. **50** (1954), 145-158.
- [O] L. O'Carroll. *Around the Eakin-Sathaye Theorem*. J. Algebra **series** **291** (2005), 259-268.
- [Sa] J. Sally. *Numbers of generators of ideals in local rings*. Marcel Dekker, Inc., New York-Basel, 1978.
- [St] R. Stanley. *Hilbert Function of graded algebra*. Advances in Math **28**, (1978), 57-83.

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