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SOME CASES OF THE EISENBUD-GREEN-HARRIS CONJECTURE

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Abstract. The Eisenbud-Green-Harris conjecture states that a homogeneous ideal in $\mathbb{k}[x_1, \ldots, x_n]$ containing a homogeneous regular sequence $f_1, \ldots, f_n$ with $\deg(f_i) = a_i$ has the same Hilbert function as an ideal containing $x_i^{a_i}$ for $1 \leq i \leq n$. In this paper we prove the Eisenbud-Green-Harris conjecture when $a_j > \sum_{i=1}^{j-1}(a_i - 1)$ for all $j > 1$. This result was independently obtained by the two authors.

1. Introduction

Let $\mathbf{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$, where $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. The following conjecture was originally made by Eisenbud, Green, and Harris [6] in the case where all the a_i are 2:

Conjecture 1 ($\mathrm{EGH}_{\mathbf{a},n}$)**.** *Let $S = \mathbb{k}[x_1, \ldots, x_n]$, where $\mathbb{k}$ is a field, and let I be a homogeneous ideal in S containing a regular sequence $f_1, \ldots, f_n$ of degrees $\deg(f_i) = a_i$. Then I has the same Hilbert function as an ideal containing $\{x_i^{a_i} : 1 \leq i \leq n\}$.*

In this paper we prove the following theorem.

Theorem 2. *The conjecture $\mathrm{EGH}_{\mathbf{a},n}$ is true if each a_j for $2 \leq j \leq n$ is larger than $\sum_{i=1}^{j-1}(a_i - 1)$.*

En-route to proving the theorem we show the equivalence of the Conjecture $\mathrm{EGH}_{\mathbf{a},n}$ with similar conjectures for regular sequences of length less than n. Firstly, if I contains a regular sequence of degrees $a_1 \leq \cdots \leq a_r$ for $r < n$, and $\mathrm{EGH}_{\mathbf{a}',n}$ holds for all $\mathbf{a}'$ with $\mathbf{a}'_i = a_i$ for $1 \leq i \leq r$ then I has the same Hilbert function as an ideal containing $x_i^{a_i}$ for $1 \leq i \leq r$. More importantly,

Proposition 3. *If $\mathrm{EGH}_{\mathbf{a},n}$ holds then every homogeneous ideal in the polynomial ring $\mathbb{k}[x_1, \ldots, x_m]$ with $m > n$ that contains a regular sequence $f_1, \ldots, f_n$ with $\deg(f_i) = a_i$ has the same Hilbert function as an ideal containing $x_i^{a_i}$ for $1 \leq i \leq n$.*

One of the original motivations of the Conjecture $\mathrm{EGH}_{\mathbf{a},n}$ was to refine the bounds given by Macaulay on the possible values of the Hilbert function $H(I, d+1)$ of an ideal I when $H(I, d)$ is specified in the case where I contains a regular sequence of degrees $\mathbf{a}$. A consequence of Proposition 3 is that any known case of $\mathrm{EGH}_{\mathbf{a},n}$ leads to a refined bound. For example, since $\mathrm{EGH}_{\mathbf{a},2}$ holds for any $\mathbf{a}$, the knowledge that an ideal I in a polynomial ring $\mathbb{k}[x_1, \ldots, x_n]$ contains a regular sequence of length two gives a smaller bound on $H(I, d+1)$ given $H(I, d)$ than that given by Macaulay,

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since the Hilbert function of such an ideal must agree with that of an ideal containing $\{x_1^{a_1}, x_2^{a_2}\}$.

2. Background

We first recall the definition of a lexicographic ideal, which will play a key role in our proof.

Definition 4. A monomial ideal $I \subseteq S$ is *lexicographic* with respect to $x_1 > x_2 > \cdots > x_n$ if whenever $x^v \succ_{lex} x^u$, with $x^u \in I$, and $\deg(x^u) = \deg(x^v)$, then $x^v \in I$. There is a unique lexicographic ideal with a given Hilbert function (see, for example, Section 4.2 of [1]).

We will need the following generalization of the notion of a lexicographic ideal, which is due to Clements and Lindström [3] (see also [8]). We use here the notation $\mathbb{N}^n_{\leq}$ for the set of all sequences $(a_1, \ldots, a_n) \in \mathbb{N}^n$ with $a_1 \leq a_2 \leq \cdots \leq a_n$. We denote such a sequence by $\mathbf{a}$.

Definition 5. Let $\succ_{lex}$ be the lexicographic order with $x_1 > x_2 > \cdots > x_n$. An ideal $I \subseteq S$ is lex-plus-powers with respect to the sequence $\mathbf{a} \in \mathbb{N}^n_{\leq}$ if $\langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle \subseteq I$, and if $x^u \in I$, with $u_i < a_i$ for $1 \leq i \leq n$, then for every x^v with $\deg(x^u) = \deg(x^v)$, $v_i < a_i$ for $1 \leq i \leq n$ and $x^v \succ_{lex} x^u$ we have $x^v \in I$.

An ideal I is lex-plus-powers if we can write $I = \langle x_1^{a_1}, \ldots, x_n^{a_n} \rangle + J$, where J is a lexicographic ideal. Note that the order on the variables in the lexicographic order is forced by the ordering of $\mathbf{a}$. Indeed, there may be no lex-plus-powers ideal with respect to another order of the variables. A simple example is given by the sequence $\mathbf{a} = (2, 3)$, and the ideal $I = \langle x^2, xy, y^4 \rangle$, which is lex-plus-powers for the order $x \succ y$. There is no ideal of the form $\langle x^2, y^3 \rangle + J$ where J is a lexicographic ideal with respect to the order $y \succ x$ with the same Hilbert function as I. We also note that this definition differs slightly from that in the work of Richert [13] and Francisco [7], as we do not require the monomials $x_i^{a_i}$ to be *minimal* generators of the lex-plus-powers ideal. See also [2], [4], [11], [9], and [12].

Clements and Lindström [3] show that for any homogeneous ideal containing the monomials $x_1^{a_1}, \ldots, x_n^{a_n}$ there is a lex-plus-powers ideal for the sequence $\mathbf{a}$ with the same Hilbert function. Thus $\mathrm{EGH}_{\mathbf{a},n}$ may be restated in an equivalent form as: if I contains a regular sequence of degrees $\mathbf{a}$ then there is lex-plus-powers ideal with respect to $\mathbf{a}$ with the same Hilbert function.

Note that by taking $a_{k+1}, \ldots, a_n$ to be arbitrarily large, we actually get the fact that if I contains $\langle x_1^{a_1}, \ldots, x_k^{a_k} \rangle$ for $k < n$, then there is a lex-plus-powers ideal containing $\langle x_1^{a_1}, \ldots, x_k^{a_k} \rangle$ with the same Hilbert function. Moreover, since the Hilbert function of a monomial ideal is independent of the base field, by extending $\Bbbk$ if necessary we can assume without loss of generality that $|\Bbbk| = \infty$.

For convenience we list the following well-known facts about regular sequences that we will use.

Lemma 6. (a) *If $J = \langle f_1, \ldots, f_r \rangle \subseteq \Bbbk[x_1, \ldots, x_n]$ is generated by a regular sequence with $\deg(f_i) = a_i$, then J has the same Hilbert function as the ideal $\langle x_1^{a_1}, \ldots, x_r^{a_r} \rangle$.*

(*b*) *A regular sequence of homogeneous elements in* $\Bbbk[x_1, \ldots, x_n]$ *remains a regular sequence after any permutation.*

(*c*) *Fix* $r < m \leq n$. *If* $f_1, \ldots, f_r$ *is a regular sequence in* $\Bbbk[x_1, \ldots, x_n]$, *with* $\Bbbk$ *infinite, and* $a_{r+1}, \ldots, a_m > 0$, *then there exist* $f_{r+1}, \ldots, f_m$ *such that* $\deg(f_i) = a_i$, *and* $f_1, \ldots, f_m$ *is a regular sequence.*

The following proposition, which is Theorem 3 of [5] (see also Corollary 5.2.19 in [10]), will also be useful.

Proposition 7. *Let* $J = \langle f_1, \ldots, f_n \rangle$ *be an ideal in* S *generated by a regular sequence with* $\deg(f_i) = a_i$. *Let* I *be an ideal containing* J *and let* $s = \sum_{i=1}^{n}(a_i - 1)$. *Then*

$$H(S/J, t) = H(S/I, t) + H(S/(J : I), s - t)$$

for $0 \leq t \leq s$.

3. Proof of the main theorem

Our approach to the main theorem will involve the following relaxation of the conjecture, where we do not assume that the regular sequence is full length.

Conjecture 8 ($\mathrm{EGH}_{n,\mathbf{a},r}$). *Let* $S = \Bbbk[x_1, \ldots, x_n]$, *and let* I *be a homogeneous ideal in* S *containing a regular sequence* $f_1, \ldots, f_r$ *of degrees* $\deg(f_i) = a_i$. *Then* I *has the same Hilbert function as an ideal containing* $\{x_i^{a_i} : 1 \leq i \leq r\}$.

Note that for $\mathrm{EGH}_{n,\mathbf{a},r}$, $\mathbf{a}$ lies in $\mathbb{N}^r_{\leq}$. In this notation $\mathrm{EGH}_{\mathbf{a},n}$ is $\mathrm{EGH}_{n,\mathbf{a},n}$. As the following two propositions show, $\mathrm{EGH}_{n,\mathbf{a},r}$ is not actually a generalization of the original conjecture $\mathrm{EGH}_{\mathbf{a},n}$.

Proposition 9. *Fix* $n > 0$ *and* $\mathbf{a} \in \mathbb{N}^r_{\leq}$. *If* $\mathrm{EGH}_{\mathbf{a}',n}$ *holds for all* $\mathbf{a}' \in \mathbb{N}^n_{\leq}$ *with* $a'_i = a_i$ *for* $1 \leq i \leq r$, *then* $\mathrm{EGH}_{n,\mathbf{a},r}$ *holds.*

Proof. Suppose that $I \subseteq S$ contains a regular sequence $f_1, \ldots, f_r$ with $\deg(f_i) = a_i$. Fix $a_{r+1} \leq a_{r+2} \leq \cdots \leq a_n$ with $a_{r+1} > a_r$, and find $f^1_{r+1}, \ldots f^1_n$ with $\deg(f^1_i) = a_i$ and $f_1, \ldots, f_r, f^1_{r+1}, \ldots, f^1_n$ a regular sequence. This is possible by Lemma 6. Let $I_1 = I + \langle f^1_{r+1}, \ldots f^1_n \rangle$. Note that $H(S/I_1, k) = H(S/I, k)$ for $1 \leq k < a_{r+1}$. Since EGH_n holds, I_1 has the same Hilbert function as an ideal containing $x_i^{a_i}$ for $1 \leq i \leq n$. Let I^1_{lex} be the lex-plus-powers ideal with respect to $x_1 > x_2 > \cdots > x_n$ with the same Hilbert function as I_1 and let K_1 be the ideal generated by those monomials in I^1_{lex} of degree less than a_{r+1}. Then K_1 also has the same Hilbert function as I in degrees less than a_{r+1}, and contains $x_1^{a_1}, \ldots, x_r^{a_r}$.

Now replace $f^1_{r+1}, \ldots, f^1_n$ by $f^2_{r+1}, \ldots, f^2_n$ so that $f_1, \ldots, f_r, f^2_{r+1}, \ldots, f^2_n$ is still regular sequence but now $\deg(f^2_i) = 2a_i$. Set $I_2 = I + \langle f^2_{r+1}, \ldots, f^2_n \rangle$, let I^2_{lex} be the lex-plus-powers ideal with respect to $x_1 > x_2 > \cdots > x_n$ containing the powers $\{x_1^{a_1}, \ldots, x_r^{a_r}, x_{r+1}^{2a_{r+1}}, \ldots, x_n^{2a_n}\}$ with the same Hilbert function as I_2, and let K_2 be the ideal generated by those monomials in I^2_{lex} of degree less than $2a_{r+1}$. Note that K_2 has the same Hilbert function as I in degrees less than $2a_{r+1}$, and contains $x_1^{a_1}, \ldots, x_r^{a_r}$. Also K_1 and K_2 agree in degrees less than a_{r+1}, since in degree k their standard monomials are the $H(S/I, k)$ lexicographically smallest monomials that are not divisible by $x_1^{a_1}, \ldots, x_r^{a_r}$. Thus $K_1 \subseteq K_2$.

We can continue in this manner, choosing $f^j_{r+1}, \ldots, f^j_n$ with $\deg(f_i) = ja_i$ completing a regular sequence. In this manner we get an increasing sequence $K_1 \subseteq K_2 \subseteq$

$K_3 \subseteq \dots$ of monomial ideals. Since S is noetherian, this sequence must eventually stabilize, so there is some N with K_j equal to K_N for all $j \geq N$. By construction K_N has the same Hilbert function as I and contains $x_1^{a_1}, \dots, x_r^{a_r}$, so is the desired ideal. □

Proposition 10. *Let* $\mathbf{a} \in \mathbb{N}^r_{\leq}$. *If* $\mathrm{EGH}_{\mathbf{a},r}$ *holds for some* r, *then* $\mathrm{EGH}_{n,\mathbf{a},r}$ *holds for all* $n \geq r$.

Proof. It suffices to show that if $\mathrm{EGH}_{n-1,\mathbf{a},r}$ holds for some $n-1 \geq r$, then $\mathrm{EGH}_{n,\mathbf{a},r}$ holds.

Suppose that $\mathrm{EGH}_{n-1,\mathbf{a},r}$ holds, and let $f_1, \dots, f_r$ be a regular sequence contained in an ideal $I \subseteq S = \Bbbk[x_1, \dots, x_{n-1}, y]$. We need to show that there is an ideal $K \subseteq S$ containing $x_1^{a_1}, \dots, x_r^{a_r}$ with the same Hilbert function as I.

By Lemma 6 we can find a linear form g such that $f_1, \dots, f_r, g$, and thus also $g, f_1, \dots, f_r$, are regular sequences. Let $N > 0$ be such that $(I : g^\infty) = (I : g^N)$.

Note that $R = S/\langle g \rangle$ is isomorphic to a polynomial ring in $n-1$ variables, and $f_1, \dots, f_r$ descend to a regular sequence $\overline{f}_1, \dots, \overline{f}_r$ in R. We will construct the desired ideal K by slices. Let $I_0 = I + \langle g \rangle$, and for $1 \leq j \leq N$ let $I_j = (I : g^j) + \langle g \rangle$. Then for $0 \leq j \leq N$ the ideal I_j regarded as an ideal of R contains $\overline{f}_1, \dots, \overline{f}_r$, so by the induction hypothesis there is an ideal in $\Bbbk[x_1, \dots, x_{n-1}]$ containing $x_1^{a_1}, \dots, x_r^{a_r}$ with the same Hilbert function as I_j. Let M_j be the lex-plus-powers ideal in $\Bbbk[x_1, \dots, x_{n-1}]$ containing $x_1^{a_1}, \dots, x_r^{a_r}$ with this Hilbert function. Let $K_j \subseteq S$ be the set of monomials $\{x^u y^j : x^u \in M_j\}$. Let $K_\infty = \{x^u y^{N+j} : j \geq 1, x^u \in M_N\}$.

Let K be the ideal generated by the monomials in $K_0, \dots, K_N$. We claim that K has the desired Hilbert function, and contains $x_i^{a_i}$ for $1 \leq i \leq r$. The latter claim is immediate, since $x_i^{a_i} \in K_0$ for $1 \leq i \leq r$. We prove the former claim by first showing that $\{x^u y^j : x^u y^j \in K\} = \cup_{j=0}^N K_j \cup K_\infty$. Note that the sets $K_0, \dots, K_N, K_\infty$ are pairwise disjoint.

To see this we show that the right-hand set is closed under multiplication, so is the set of monomials in a monomial ideal. It thus suffices to show that if $x^u y^j$ is not in the right-hand set, then neither is any monomial of the form $x^u y^{j-1}$ or $x^u y^j / x_i$. The latter is immediate from the definition of K_j, as if $x^u y^j \notin K_j$, with $j \leq N$, then $x^u \notin M_j$, so $x^u / x_i \notin M_j$, and so $x^u y^j / x_i \notin K_j$. If $j > N$, then $x^u y^j \notin K_\infty$ means that $x^u \notin M_N$, and so $x^u / x_i \notin M_N$, and thus $x^u y^j / x_i \notin K_\infty$. To see the former claim, we note that since $(I : g^{j-1}) \subseteq (I : g^j)$, we have $I_{j-1} \subseteq I_j$, and thus $M_{j-1} \subseteq M_j$. So if $j \leq N$ and $x^u y^j \notin K_j$, then $x^u \notin M_j$, and thus $x^u \notin M_{j-1}$, so $x^u y^{j-1} \notin K_{j-1}$. If $j > N$, then $x^u y^j \notin K_\infty$ means that $x^u \notin K_N$, so $x^u \notin M_N$, and thus $x^u y^{j-1} \notin K_N \cup K_\infty$. This shows that the right-hand side set is the set of monomials in a monomial ideal, and since K is by definition the monomial ideal generated by these monomials, we have the equality.

We finish by showing that K has the correct Hilbert function. Recall that $(I : g^j) = (I : g^N)$ for $j \geq N$, so $(I : g^j) + \langle g \rangle = I_N$ for such j. Note that the number of monomials $x^u y^j$ not in K_j (or K_∞ if $j > N$) of degree t is the number of monomials x^u of degree $t-j$ not in M_j (or M_N), so we have the following formula for the Hilbert function of S/K:

$$\begin{aligned} H(S/K,t) &= \sum_{j=0}^{N} H(S/M_j,t-j) + \sum_{j=N+1}^{t} H(S/M_N,t-j) \\ &= \sum_{j=0}^{N} H(S/I_j,t-j) + \sum_{j=N+1}^{t} H(S/I_N,t-j) \\ &= \sum_{j=0}^{t} H(S/((I:g^j),g),t-j). \end{aligned}$$

By considering the short exact sequence

$$0 \to S/(J:g) \longrightarrow S/J \longrightarrow S/(J,g) \to 0,$$

we see that for any ideal J we have $H(S/J,t) = H(S/(J:g),t-1) + H(S/(J,g),t)$. Applying this repeatedly we see that $H(S/I,t) = \sum_{j=0}^{t} H(S/((I:g^j),g),t-j) = H(S/K,t)$, completing the proof. □

Conjecture 1 can be thought as a conjecture on the growth of ideals containing a regular sequence of given degrees. More precisely let I be a homogeneous ideal containing a regular sequence of degrees given by the nondecreasing sequence $\mathbf{a}$ and let d be a non-negative integer. Note that there exists a unique lex-plus-power ideal J of the form $J = \langle x_1^{a_1},\dots,x_n^{a_n}\rangle + L$, where L is a lexicographic ideal generated by monomials of the same degree d, with the property that $H(I,d) = H(J,d)$. It is known that $\mathrm{EGH}_{\mathbf{a},n}$ holds if and only if for any I, d and J, defined as above, the condition $H(I,d+1) \geq H(J,d+1)$ is also satisfied. We therefore specialize the conjecture at any single degree in the following way.

Definition 11. Let d be a non-negative integer. We say that $\mathrm{EGH}_{\mathbf{a},n}(d)$ holds if for any homogeneous ideal $I \subseteq \Bbbk[x_1,\dots,x_n]$ containing a regular sequence of degrees $\mathbf{a} \in \mathbb{N}^n_{\leq}$ there exists a homogeneous ideal J containing $\{x_i^{a_i} : 1 \leq i \leq r\}$ such that $H(I,d) = H(J,d)$ and $H(I,d+1) = H(J,d+1)$.

Note that by the Clements-Lindström Theorem [3] we can assume that J is a lex-plus-powers ideal with respect to $\mathbf{a}$. Thus $\mathrm{EGH}_{\mathbf{a},n}$ holds if and only if $\mathrm{EGH}_{\mathbf{a},n}(d)$ holds for all non-negative integers d. The following Lemma shows a symmetry in Conjecture 1.

Lemma 12. *Let $\mathbf{a} \in \mathbb{N}^n_{\leq}$ and let $s = \sum_{i=1}^{n}(a_i - 1)$. Then for a non-negative integer d we have that $\mathrm{EGH}_{\mathbf{a},n}(d)$ holds if and only if $\mathrm{EGH}_{\mathbf{a},n}(s-d-1)$ holds.*

Proof. Assume $\mathrm{EGH}_{\mathbf{a},n}(d)$ holds. Let I be a homogeneous ideal of $S = \Bbbk[x_1,\dots,x_n]$ containing a regular sequence $f_1,\dots,f_n$ of degrees $\mathbf{a} \in \mathbb{N}^n_{\leq}$. Let $F = \langle f_1,\dots,f_n\rangle$ and let $I_1 = (F:I)$. By $\mathrm{EGH}_{\mathbf{a},n}(d)$ there exists an ideal J containing $M = \langle x_1^{a_1},\dots,x_n^{a_n}\rangle$ such that $H(I_1,d) = H(J,d)$ and $H(I_1,d+1) = H(J,d+1)$. Set J_1 equal to $(M:J)$. Note J_1 contains M so by Lemma 6 and Proposition 7 we have $H(I,s-d) = H(J_1,s-d)$ and $H(I,s-d-1) = H(J_1,s-d-1)$. □

Theorem 13. *The conjecture $\mathrm{EGH}_{\mathbf{a},n}$ is true if $a_j > \sum_{i=1}^{j-1}(a_j - 1)$ for all $j > 1$.*

Proof. The proof is by induction on n. When $n = 1$ the hypothesis $\mathrm{EGH}_{a,1}$ states that if a homogeneous ideal $I \subset \Bbbk[x_1]$ contains an element f of degree a_1 then it has the same Hilbert function as an ideal containing $x_1^{a_1}$. This is immediate, since $x_1^{a_1}$ is the only homogeneous polynomial of degree a_1 in $\Bbbk[x_1]$. We now assume that $\mathrm{EGH}_{\mathbf{a}',n-1}$ holds, where $\mathbf{a}' \in \mathbb{N}^{n-1}_{\leq}$ is the projection of $\mathbf{a}$ onto the first $n-1$ coordinates. Let $s = \sum_{i=1}^{n}(a_i - 1)$. By Lemma 12 it is enough to prove $\mathrm{EGH}_{\mathbf{a},n}(d)$ for $0 \leq d \leq \lfloor (s-1)/2 \rfloor$. Let $I \subseteq \Bbbk[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of degrees $\mathbf{a}$. By Proposition 10 we have that $\mathrm{EGH}_{n,\mathbf{a}',n-1}$ holds, so there is an ideal J, containing $x_1^{a_1}, \ldots, x_{n-1}^{a_{n-1}}$ with the same Hilbert function as I. Since, by assumption, $a_n > \lfloor (s-1)/2 \rfloor + 1$ we deduce that $H(I,d) = H(J + (x_n^{a_n}), d)$ for all $0 \leq d \leq \lfloor (s-1)/2 \rfloor + 1$. □

Remark 14. Note that Theorem 13 implies $\mathrm{EGH}_{\mathbf{a},2}$ for any $\mathbf{a} = (a_1, a_2)$, which was already known.

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