BÜCHI’S 5-SQUARE PROBLEM

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A Büchi $n$-tuple $x_1 < x_2 < \cdots < x_n$ of nonnegative integers is a solution of the system

$$x_1^2 - 2x_2^2 + x_3^2 = x_2^2 - 2x_3^2 + x_4^2 = \cdots = x_{n-2}^2 - 2x_{n-1}^2 + x_n^2 = 2,$$

not of the “trivial” form $(m, m+1, \ldots, m+n-1)$ ($m \in \mathbb{N}$).

The case $n = 3$ is elementary (there is a formula giving all solutions), and there are infinitely many Büchi 4-tuples (can you find even one?), but no one knows whether or not Büchi 5-tuples exist. Büchi raised this question in the 1970s after noting that a negative answer—even when 5 is replaced by some larger $n$—would lead from the theorem of Matiyesevich to nonexistence of an algorithm for deciding whether any diagonal diophantine system $\sum a_{ij}x_j^2 = b_i$ has a solution. (To illustrate: if there were such an algorithm, then one could answer—which no one has—the question of whether there exists a rectangular parallelepiped for which any line segment joining two vertices has integer length.)

Work of Vojta on connections with a conjecture of Lang on surfaces of general type will be touched on, as well as an unproved conjecture on Büchi 4-tuples that holds for $x_2 < 10^{500}$. 