Spetember 22, 2016

MATH 450

Midterm Exam 1

Solutions

Instructions: Give a complete solution to each problem. You may use any result from class, the book, or homework **except** the statement you are asked to prove. You may also use any fact established in Calculus or Linear Algebra classes. Be sure to justify your statements.

1. **(10 points)** Use the Euclidean algorithm to compute $d = (1416, 456)$, and write d as an integer linear combination of 1416 and 456.

   Solution: Note,

$$1416 = 3 \cdot 456 + 48;$$

$$456 = 9 \cdot 48 + 24;$$

$$48 = 2 \cdot 24 + 0,$$

   So,

$$d = 24 = 456 - 9 \cdot 48 = 456 - 9 \cdot (1416 - 3 \cdot 456) = 28 \cdot 456 - 9 \cdot 48.$$

$\square$

2. **(12 points)** Let G be a group and suppose H is a non-empty subset of G. Show that H is a subgroup of G if and only if $xy^{-1} \in H$ for all $x, y \in H$.

   Solution: Suppose H is a subgroup. Let $x, y \in H$. Then, since H is closed under inversion, $y^{-1} \in H$. Now since $x, y^{-1} \in H$ and H is closed under the group operation, we have $xy^{-1} \in H$.

   Now, suppose $H \neq \emptyset$, and $xy^{-1} \in H$, for all $x, y \in H$. We need to show H is closed under the group operation and inversion. Since $H \neq \emptyset$, there is some $x \in H$. Therefore, $e = xx^{-1} \in H$ by our assumption. Now, $ex^{-1} = x^{-1} \in H$, from our

assumption, for any $x \in H$. Thus, H is closed under inversion. Finally, if $x, y \in H$, then we have just seen $y^{-1} \in H$, and so, by our assumption $x(y^{-1})^{-1} = xy \in H$. Thus, by the subgroup criterion, H is a subgroup of H. $\square$

3. **(13 points)** Let G be a group and suppose that x and y are elements of G. Prove, using some form of induction, that, for all $n \geq 0$, $(xyx^{-1})^n = xy^nx^{-1}$. Be sure to be explicit about where you use the induction hypothesis.

   Solution: We note first that for $n = 0$, we have $(xyx^{-1})^0 = e = xex^{-1} = xy^0x^{-1}$. So the claim holds when $n = 0$. Now suppose, for some $n \geq 0$, we have $(xyx^{-1})^n = xy^nx^{-1}$. Then,

$$(xyx^{-1})^{n+1} = (xyx^{-1})^n(xyx^{-1}) = (xy^nx^{-1})(xyx^{-1}),$$

   by our inductive hypothesis. Now, we have

$$(xyx^{-1})^{n+1} = (xy^nx^{-1})(xyx^{-1}) = xy^n(x^{-1}x)yx^{-1} = xy^neyx^{-1} = xy^{n+1}x^{-1}.$$

   Thus, if the claim holds for n, it also holds for $n + 1$. Therefore, the claim holds for all $n \geq 0$ by induction. $\square$

4. **(5 points each)** In each of the following cases, determine whether or not H is a subgroup of G. Give an explanation for your answer.

   (a) Let $G = \mathbb{Z}$, and $H = \{n \mid n \equiv 3 \pmod 7\}$.

   (b) Let $G = GL(2, \mathbb{R})$ and $H = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in G \middle| \; ac = 1 \right\}$.

   (c) Let $G = U(13)$ and $H = \{1, 5, 8, 12\}$.

   Solution:

   (a) The identity of $\mathbb{Z}$ is 0, and $0 \notin H$, so H cannot be a subgroup.

   (b) Note, if $X = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}, Y = \begin{pmatrix} x & y \\ 0 & c \end{pmatrix} \in H$, then

$$XY^{-1} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} z & -y \\ 0 & x \end{pmatrix} = \begin{pmatrix} az & bx - ay \\ 0 & cx \end{pmatrix},$$

and $(az)(cx) = (ac)(xz) = 1 \cdot 1 = 1$, so $XY^{-1} \in H$. Thus, by problem 2, H is a subgroup.

(c) Note H is closed under the group operation: $x \cdot y = y \cdot x$, for all $x, y \in H$, $1 \cdot x = x$, for all x, and,

$$5^2 = 12, 5 \cdot 8 = 1, 5 \cdot 12 = 8, 8^2 = 12, 8 \cdot 12 = 5, \text{ and } 12^2 = 1.$$

Since H is closed and finite it is a subgroup. $\square$

5. True/False (5 points each) Determine whether each of the following statements is true or false. If true, give a proof. If false, give a concrete counterexample.

(a) If G is abelian and H is a subgroup of G then H is abelian.

(b) If G is an non-abelian group, then every proper subgroup of G is non-abelian.

(c) If G is a finite group, and m divides the order of G then G has an element of order m.

(d) Any two groups of a given order are isomorphic.

(e) The center $Z(G)$ of any group G is a normal subgroup of G.

Solution:

(a) **True:** If $x, y \in H$ then, since the operation in G is commutative, $xy = yx$ in both G and H, as H inherits the G operation. Thus, H is abelian.

(b) **False:** Let G be the nonabelian group D_4, and $H = \{e, R_{180}\}$, where R_{180} is rotation through $180°$ in the clockwise direction. Then H is an abelian subgroup.

(c) **False:** Let $G = S_3$, and $m = 6$. Then m divides $|G|$ but there is no element of order 6 in G.

(d) **False:** Let $G_1 = U_5$ and $G_2 = U_8$. Both have order 4. Note G_1 is cyclic, and G_2 is not. So they are not isomorphic.

(e) **True:** If $x \in G$ and $z \in Z(G)$ then $zx = xz$, so $Z(G)x = xZ(G)$, and thus $Z(G) \triangleleft G$.

6. **(15 points)** Prove that a group G is abelian if and only if the map $f : G \to G$ given by $f(x) = x^{-1}$ is an isomorphism.

Solution: Suppose G is abelian. Then $ab = ba$ for all $a, b \in G$. Thus, for all $a, b \in G$, we have $f(ab) = (ab)^{-1} = b^{-1}a^{-1} == a^{-1}b^{-1} = f(a)f(b)$. Thus, f is a homomorphism. Moreover, $f(x) = e$ if and only if $x^{-1} = e$, which is equivalent to $x = e$, so $\operatorname{Ker} f = \{e\}$, which says f is a monomorphism. Also, if $x \in G$, then $x = (x^{-1})^{-1} = f(x^{-1})$, so f is sujrective as well. Thus, f is an isomorphism.

On the other hand, suppose f is an isomorphism. Let $a, b \in G$. Then $ba = (a^{-1}b^{-1})^{-1} = f(a^{-1}b^{-1}) = f(a^{-1})f(b^{-1}) = ab$, so G is abelian. $\square$

7. **(5 points each)** Recall $GL_n(\mathbb{R}) = \{n \times n \text{ invertible matrices with real entries}\}$ is a group with the operation of matrix multiplication, and $\mathbb{R}^\times = \{x \in \mathbb{R} | x \neq 0\}$ is a group with the operation of ordinary multiplication of real numbers. (You don't need to prove any of this).

   (a) Prove that the determinant map, $\det : GL_n(\mathbb{R}) \to \mathbb{R}^\times$ is a homomorphism. What is the kernel?

   (b) Construct a monomorphism $\varphi : \mathbb{R}^\times \to GL_2(\mathbb{R})$.

   Solution:

   (a) We know from linear algebra that, for $A, B \in GL_n(\mathbb{R})$ we have $\det(AB) = \det A \det B$, so det is a homomorphism.

   (b) Let $\varphi(r) = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix}$. Note

$$\varphi(rs) = \begin{pmatrix} rs & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix},$$

   so φ is a homomorphism. Also, note $\varphi(s) = \varphi(r)$ if and only if

$$\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 1 \end{pmatrix} \Leftrightarrow s = r,$$

   so φ is a monomorphism. $\square$