November 3, 2016

MATH 450

Midterm Exam 2

Solutions

Instructions: Give a complete solution to each problem. You may use any result from class, the book, or homework **except** the statement you are asked to prove. You may also use any fact established in Calculus or Linear Algebra classes. Be sure to justify your statements.

Notation:

$\mathrm{Syl}_p(G) = \{P \subset G | P \text{ is a Sylow } p-\text{subgroup of } G\}$

$n_p(G) = |\,\mathrm{Syl}_p(G)|$.

If $\sigma \in S_n$ and $\sigma = (a_{11}a_{12}\cdots a_{1m_1})(a_{21}a_{22}\cdots a_{2m_2})\cdots(a_{r1}a_{r2}\cdots a_{rm_r})$, as a product of disjoint cycles, then we say σ has **cycle type** $m_1, m_2, \ldots m_r$.

1. **(5 points)** Let G be a group and $N \triangleleft G$. Let $\bar{G} = G/N$, and for $x \in G$ denote by $\bar{x}$ the element $xN \in \bar{G}$. Show that $\bar{x}$ and $\bar{y}$ commute in $\bar{G}$ if and only if $xyx^{-1}y^{-1} \in N$.

   Solution: By definition of the product in G/N, we have $\bar{x}\bar{y} = \overline{xy}$. So,

$$\bar{x}\bar{y} = \bar{y}\bar{x} \text{ if and only if } \overline{xy} = \overline{yx},$$

$$\text{if and only if } (xy)(yx)^{-1} \in N \text{ if and only if } xyx^{-1}y^{-1} \in N. \quad \square$$

2. **(4 points each)** Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 4 & 6 & 8 & 5 & 7 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 2 & 5 & 4 & 8 & 7 & 6 \end{pmatrix}$.

   (a) Write $\alpha\beta$ as a product of disjoint cycles.

   (b) Write $\alpha\beta$ as a product of transpositions.

   (c) Compute α^{-1}.

   Solution: By direct computation, we get,

   (a) $\alpha\beta = (132)(4675)$.

   (b) $\alpha\beta = (12)(13)(45)(47)(46)$.

(c)

$$\alpha^{-1} = \begin{pmatrix} 3 & 2 & 1 & 4 & 6 & 8 & 5 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 3 & 2 & 1 & 4 & 7 & 5 & 8 & 6 \end{pmatrix}.$$

3. **True/False (6 points each)** Determine whether each of the following statements is true or false. If true, give a proof. If false, give a concrete counterexample.
   (a) If H and K are subgroups of a group G, with $H \triangleleft K$ and $K \triangleleft G$, then $H \triangleleft G$.
   (b) If H is a subgroup of $G_1 \times G_2$, then $H = H_1 \times H_2$, for some subgroups H_1 of G_1 and H_2 of G_2.
   (c) If α is an odd permutation, then α^{-1} is an odd permutation.

   Solution:
   (a) **False** Let $G = D_4$, $K = \{1, (12)(34), (13)(24), (14)(23)\}$, and $H = \{1, (12)(34)\}$. Note $K \triangleleft G$, since$K$ is of index 2 in G, and similarly $H \triangleleft K$, but $(1234)(12)(34)(1234)^{-1} = (1234)((12)(34)(1432) = (14)(23 \notin H$, so H is not normal in G.
   (b) **False** Let $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, and $H = \{(0,0),(1,1)\}$. Then we cannot find H_1, and H_2 with $H \simeq H_1 \times H_2$.
   (c) **True** We have $e = \alpha\alpha^{-1}$, an since e is an even permutation, and α is odd, then α^{-1} is alos odd.
4. (a) **(9 points)** Prove that if $\sigma, \tau \in S_n$ have the same cycle type, then there is some $\rho \in S_n$ with $\tau = \rho\sigma\rho^{-1}$.
   (b) **(11 points)** What is the order of the centralizer, $C_{S_n}((123 \cdots n)) = C((123 \cdots n))$ of $(123 \cdots n)$ in S_n?

Solution

(a) Let

$$\sigma = (a_{11}a_{12}\ldots a_{1m_1})(a_{21}a_{22}\ldots a_{2m_2})\cdots(a_{r1}a_{r2}\ldots a_{rm_r})$$

and

$$\tau = (b_{11}b_{12}\ldots b_{1m_1})(b_{21}b_{22}\ldots b_{2m_2})\cdots(b_{r1}b_{r2}\ldots b_{rm_r})$$

Let $S = \{a_{ij} | 1 \le i \le r, 1 \le j \le m_i\}$ and $T = \{b_{ij} | 1 \le i \le r, 1 \le j \le m_i\}$. Note, $|S| = |T|$. Let $k = |S|$. Let $S' = \{1, 2, \ldots, n\} \setminus S = \{1 \le i \le n | i \notin S\}$ and similarly, let $T' = \{1, 2, \ldots, n\} \setminus T$. Note, $|S'| = |T'| = n - k$. Fix any one to one (and hence onto) function $\rho' : S' \to T'$. Let

$$\rho(s) = \begin{cases} b_{ij} & \text{if } s = a_{ij}, \\ \rho'(s) & \text{if } s \notin S. \end{cases}$$

Note, if $\rho(s) = \rho(t)$, and $s \in S$, then $\rho(s) \in T$, so $\rho(t) \in T$, and so $s = t$, since the cycles of σ and τ are disjoint. If $s \in S'$, then $\rho(s) = \rho(t) \in T'$, and by the definition of ρ' we also have $s = t$. Thus, ρ is one-to-one, and hence also onto. Now if $b_{ij} \in T$, then

$$\rho\sigma\rho^{-1}(b_{ij}) = \rho(\sigma(a_{ij})) = \rho(a_{i(j+1)}) = b_{i(j+1)}, \tag{1}$$

with $j+1$ taken modulo m_i. If $t \in T'$, then $s = \rho^{-1}(t) \in S'$, so $\sigma(s) = s$. Thus,

$$\rho\sigma\rho^{-1}(t) = \rho(\sigma(s)) = \rho(s) = t. \tag{2}$$

Then, (1) and (2) show $\rho\sigma\rho^{-1} = \tau$.

(b) By (a) the conjugacy class $\mathcal{C}$ of $(12\cdots n)$ is the set of all n–cycles in S_n, i.e.,

$$\mathcal{C} = \{(a_1a_2\dots a_n) \in S_n\}.$$

Now note

$$(a_1a_2\dots a_n) = (a_2a_3\dots a_na_1) = \cdots = (a_na_1\dots a_{n-1}).$$

Thus

$$|\mathcal{C}| = \frac{n!}{n} = (n-1)!.$$

By the Orbit-Stabilizer Theorem,

$$|\mathcal{C}| = \frac{|S_n|}{|C((12\dots n))|}.$$

So,

$$(n-1)! = \frac{n!}{|C((12\dots n))|},$$

Which shows $|C(12\dots n)| = n$. □

Remark: Note this shows that an n–cycle only commutes with its powers.

5. **(8 points each)**

(a) Show that a group of order 56 must have a normal Sylow p–subgroup for some p.

(b) Let G be a finite group, and suppose Q is a normal p–subgroup of G. Show that if P is a Sylow p–subgroup, then $Q \subset P$.

Solution:

(a) By Sylow II, a Sylow p–subgroup P is normal in G if and only if it is the unique Sylow p–subgroup, i.e., if and only if $n_p = 1$. By Sylow III we have $n_p \equiv 1(\text{mod} p)$ and $n_p \big| (|G|/|P|)$. Since $56 = 2^3 \cdot 7$, we have $n_7 = 1$ or 8. Suppose $n_7 = 8$, $P, Q \in \text{Syl}_7(G)$, with $P \neq Q$. Since $|P| = |Q| = 7$, we see $P \cap Q = \{e\}$. Thus, there are $8 \cdot 6 = 48$ elements of order 7 in G. Thus, there are at most 8 elements in G with order dividing 8. Since there must be 8 such elements in any Sylow 2-subgroup, we see $n_2 = 1$. So either $n_7 = 1$ or $n_2 = 1$, and thus G always has a normal Sylow subgroup.

(b) By the proof of Sylow II, we know $Q \subset P$, for some $P \in \mathrm{Syl}_p(G)$. Now, if $P' \in \mathrm{Syl}_p(G)$, we have $P' = xPx^{-1}$ for some $x \in G$ by Sylow II. Since $Q \triangleleft G$, we have $xQx^{-1} = Q$. Thus, $Q = xQx^{-1} \subset xPx^{-1} = P'$. So $Q \subset P'$ for all $P' \in \mathrm{Syl}_p(G)$. $\square$

6. (a) **(4 points)** State the Fundamental Theorem of Finite Abelian Groups

   (b) **(8 points)** List all the isomorphism classes of abelian groups of order 200.

Solution:

(a) **Theorem: The Fundamental Theorem of Finite Abelian Groups** Any finite abelian group is a direct product of cyclic groups. In particular, any finite abelian group is a direct product of cyclic groups of prime power order.

(b) Since $200 = 2^3 5^2$, we know an abelian group of order 200 is isomorphic to one of the following:

i) $G_1 = \mathbb{Z}_8 \times \mathbb{Z}_{25} \simeq \mathbb{Z}_{200}$;

ii) $G_2 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$;

iii) $G_3 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{25}$;

iv) $G_4 = \mathbb{Z}_8 \times \mathbb{Z}_5 \times \mathbb{Z}_5$;

v) $G_5 = \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$;

vi) $G_6 = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_5$.

7. **(17 points)** State the First Homomorphism Theorem (also known as the First Isomorphism Theorem).

Theorem: If $\varphi : G \to G'$ is a homomorphism, with kernel K, then $\varphi(G) \simeq G/K$, with the isomorphism given by $\psi(Ka) = \varphi(a)$.

Extra Credit: (10 points) Suppose $P \in \mathrm{Syl}_p(G)$, and $H = N_G(P)$. Prove $N_G(H) = H$.

Solution:

Note, since $P \subset H$, we have $|P| \big| |H|$. So, if $|G| = p^n m$, with $p \not| m$, then $|H| = p^n k$, with $p \not| k$. Thus, $P \in \mathrm{Syl}_p(H)$. Also, note, by definition, $H = \{x | xPx^{-1} = P\}$, and so $P \triangleleft H$. Thus, $n_p(H) = 1$. Now, if $g \in N_G(H)$, then $gHg^{-1} = H$. Thus, $gPg^{-1} \subset H$, and so $gPg^{-1} \in \mathrm{Syl}_p(H)$. Therefore, by what we said above, $gPg^{-1} = P$. Therefore, $g \in N_G(P) = H$. Thus, $N_G(H) = H$. $\square$