R-groups and Elliptic Representations for SL_n

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Abstract. We determine the reducibility and number of components of any representation of $SL_n(F)$ which is parabolically induced from a discrete series representation. The R-groups are computed in terms of restriction from $GL_n(F)$, extending the results of Gelbart and Knapp. This yields an explicit description of the elliptic tempered representations of $SL_n(F)$. We also describe those tempered representations which are not irreducibly induced from elliptic representations.

To Elizabeth:

Introduction. We continue our investigation of those representations of classical p-adic groups which are parabolically induced from the discrete series. We now consider the group $G = SL_n(F)$. We will describe explicit criteria for reducibility of induced representations, determine the number of constituents of such representations, and develop criteria for the constituents to be elliptic. Moreover, we can describe those irreducible tempered representations of G which are not elliptic, and are also not irreducibly induced from an elliptic representation.

We use the technique of restriction from $\tilde{G} = GL_n(F)$. This technique has been used by several authors to describe various aspects of the representation theory of G [**4**,**5**,**6**,**7**,**14**,**19**,**20**,**21**,**22**,**24**,**30**]. Our purpose here is to use some of these results to obtain information on the structure of the generalized principal series for G.

Let $P = MN$ be a parabolic subgroup of G. Suppose that σ is an irreducible discrete series representation of M. We wish to determine when the unitarily induced representation $i_{G,M}(\sigma)$ is reducible, and if so, what is the structure of its components. We use the theory of R-groups, as developed by Knapp and Stein [**18**], and Silberger [**28**]. This, along with the multiplicity one result of Howe and Silberger [**14**], determines the structure of the commuting algebra $C(\sigma)$.

The R-group is a quotient of the subgroup, $W(\sigma)$, of Weyl group elements which fix σ. If Δ' is the collection of reduced roots for which the Plancherel measure of σ vanishes, then $R \simeq W(\sigma)/W'$, where W' is the group generated by reflections in the roots in Δ'.

For the groups $Sp_{2n}(F)$, $SO_n(F)$, and $U_n(F)$, we were able to explicitly describe the group $W(\sigma)$, and use the properties of Plancherel measures to determine which groups could possibly arise as R-groups [**9**,**10**]. However, what precise R-groups can arise has yet to be determined, since the explicit computation of Plancherel measures is not completed in these cases. The R-groups for certain parabolics are understood completely [**8**,**27**]. In the case of SL_n, the Plancherel measures are well understood [**24**,**25**]. Moreover, there is already a necessary condition, in terms of restriction, for a Weyl group element w to be in $W(\sigma)$ [**24**]. We show that this condition is sufficient, and thus we obtain an explicit description for the R-group, where all the pieces are understood.

Let $\tilde{P} = \tilde{M}N$ be a parabolic of $\tilde{G}$, with $P = \tilde{P} \cap G$, and $M = \tilde{M} \cap G$. Then there is a discrete series representation, π_σ, of $\tilde{M}$ so that $\pi_\sigma|_M$ contains σ as a constituent. The components of $\pi_\sigma|_M$ are said to be L-indistinguishable. Since $i_{G,M}(\sigma) \hookrightarrow i_{\tilde{G},\tilde{M}}(\pi_\sigma)$, the Plancherel measures for σ are the same as those for π_σ [**24**]. The reducibility of induced representations for GL_n are well understood [**3**,**23**], and we know the Plancherel measures for π_σ explicitly [**25**]. Therefore, we know the zeros of the Plancherel measures for σ by restriction. We then show that $w \in W(\sigma)$ if and only if $w\pi_\sigma \simeq \pi_\sigma \otimes \eta \circ \det$, for some $\eta \in \hat{F}$ (cf. Lemma 2.3). A lemma of Shahidi [**24**] shows that W' is the set of w with the property that $w\pi_\sigma \simeq \pi_\sigma$. This gives an explicit description of R, as a group of characters, and generalizes the results of [**7**]. For a fixed η, we construct a unique element, w_η, with $w_\eta \in R$, and $w_\eta\pi_\sigma \simeq \pi_\sigma \otimes \eta \det$ (cf. Theorem 2.6). We use this explicit description of the elements of R, and a theorem of Arthur [**1**], to describe the elliptic tempered representations of G (cf. Theorem 3.4). We also give an explicit description of those irreducible tempered representations of G which are not of the form $i_{G,M'}(\tau)$ for some Levi subgroup M', and some elliptic representation τ of M' (cf. Theorem 3.8). This is based on a result of Herb [**13**].

Many results on reducibility and number of components are also obtainable by the method of Hecke algebra isomorphisms. Thus, our reducibility results should match those in forthcoming work of Bushnell and Kutzko [**5**].

I would like to thank Rebecca Herb, Stephen Kudla, Phil Kutzko, Paul Sally, Freydoon

Shahidi, and Marko Tadic for comments and conversations which furthered the results herein. I would like to thank Anthony Knapp and the referee for pointing out some oversights on my part.

§**1 Preliminaries.** Let F be a locally compact, non–discrete, nonarchimedean field of characteristic zero. Let q be the residual characteristic of F. Let $\mathbf{G}$ be a connected reductive quasi-split algebraic group defined over F. Let G be the F-rational points of $\mathbf{G}$. We say that an element x of G is elliptic if its centralizer is compact, modulo the center of G. We let G^e denote the set of regular elliptic elements of G [**12**].

Let $\mathcal{E}_2(G)$ denote the collection of equivalence classes of irreducible discrete series representations of G, and denote by $\mathcal{E}_t(G)$ the equivalence classes of irreducible tempered representations of G. Then $\mathcal{E}_2(G) \subset \mathcal{E}_t(G)$. If $\pi \in \mathcal{E}_t(G)$, then we denote its character by Θ_π. Since Θ_π can be viewed as a locally integrable function [**11**], we can consider its restriction to G^e, which we denote by Θ_π^e. We say that π is elliptic if $\Theta_\pi^e \neq 0$. In general, we would like to describe $\mathcal{E}_t(G)$, and explicitly determine which classes are elliptic.

We say that $M \subseteq G$ is a Levi subgroup of G if there is a parabolic subgroup P of G with M as its Levi component. Let N be the unipotent radical of P. If A_0 is a maximal F-split torus of G, then we let $\Phi(G, A_0)$ be the set of roots of A_0 in G. Let Δ be a collection of simple roots. Then the conjugacy classes of parabolic subgroups of G are in one to one correspondence with subsets of Δ. If $\theta \subset \Delta$, then we let A_θ be the subtorus of A_0 corresponding to θ. Let $B = TU$ be the Borel subgroup associated to $A_\emptyset = A_0$. Then a Levi subgroup M is called standard if there is a parabolic $P = MN$, with $P \supset B$. In this case, P is also called standard.

If M is a Levi subgroup with split component A, then we denote the Weyl group $N_G(A)/Z_G(A)$ by $W(G/A)$ or $W(A)$. Let $\widetilde{w} \in W(A)$, and choose a representative w for $\widetilde{w}$ in $N_G(A)$. If (σ, V) is an irreducible tempered representation of M, then we let $\widetilde{w}\sigma$ be the representation defined by the formula $\widetilde{w}\sigma(m) = \sigma(w^{-1}mw)$. The class of $\widetilde{w}\sigma$ is independent of the choice of w. We say that σ is ramified if there is some non–trivial $\widetilde{w} \in W(A)$, with $\widetilde{w}\sigma \simeq \sigma$. We denote by $Ind_P^G(\sigma)$ the representation unitarily induced by σ. Since its class depends only on M, not P, we may also denote it by $i_{G,M}(\sigma)$.

We denote by $X(M)_F$ the collection of F-rational characters of M. We let $\mathfrak{a} = \mathrm{Hom}(X(M)_F, \mathbb{Z})$, be the real Lie algebra of A, and let $\mathfrak{a}^*_{\mathbb{C}}$ be the complexified dual of $\mathfrak{a}$ [**12**]. Then there is a homomorphism $H_P\colon M \to \mathfrak{a}$ which satisfies

$$q^{<\chi, H_P(m)>} = |\chi(M)|_F, \ \forall\, \chi \in X(M)_F, \ m \in M.$$

For any $\nu \in \mathfrak{a}^*_{\mathbb{C}}$ and $\sigma \in \mathcal{E}_2(M)$, we let

$$I(\nu, \sigma) = Ind_P^G \left(\sigma \otimes q^{<\nu, H_P()>} \right).$$

The space $V(\nu, \sigma)$ of $I(\nu, \sigma)$ is given by

$$V(\nu, \sigma) = \left\{ f : G \to V \middle| f(mng) = \delta_P^{1/2}(m)\sigma(m) q^{<\nu, H_P(m)>} f(g), \ \forall g \in G, m \in M, n \in N \right\}.$$

Here δ_P is the modular function of P. If $\widetilde{w} \in W(A)$, then we let $N_{\widetilde{w}} = U \cap w^{-1}\overline{N}w$, where $\overline{N}$ is the unipotent radical opposed to N. We formally define an operator on $V(\nu, \sigma)$ by

$$A(\nu, \sigma, w) f(g) = \int_{N_{\widetilde{w}}} f(w^{-1}ng)dn.$$

If the integral converges for every choice of f and g, then we say that $A(\nu, \sigma, w)$ converges. If $A(\nu, \sigma, w)$ converges then it defines an intertwining operator between $I(\nu, \sigma)$ and $I(w\nu, w\sigma)$.

Theorem 1.1 (Harish-Chandra). *Let $\widetilde{w} \in W(A)$ and $\sigma \in \mathcal{E}_2(M)$. Let P' be the standard parabolic subgroup with Levi component $w^{-1}Mw$. Then $A(\nu, \sigma, w)$ converges for ν in the positive Weyl chamber, and can be extended to a meromorphic function of ν on $\mathfrak{a}^*_{\mathbb{C}}$. Moreover, there is a complex number $\mu(\nu, \sigma, \widetilde{w})$ so that*

$$A(\nu, \sigma, w)A(w\nu, w\sigma, w^{-1}) = \mu(\nu, \sigma, \widetilde{w})^{-1} \gamma_{\widetilde{w}}(G/P)\gamma_{\widetilde{w}^{-1}}(G/P'),$$

where the constant $\gamma_{\widetilde{w}}(G/P)$ is defined in [**12**]. *Moreover, $\nu \to \mu(\nu, \sigma, \widetilde{w})$ is meromorphic on $\mathfrak{a}^*_{\mathbb{C}}$, and holomorphic on $i\mathfrak{a}^*$.* □

The factor $\mu(\nu, \sigma, \widetilde{w})$ is called the Plancherel measure associated to ν, σ and $\widetilde{w}$. When $\widetilde{w}$ is the longest element of the Weyl group, we write $\mu(\nu, \sigma) = \mu(\nu, \sigma, \widetilde{w})$, and write

$\mu(\sigma) = \mu(0, \sigma)$. If M is a maximal proper Levi subgroup, then $i_{G,M}(\sigma)$ is reducible if and only if σ is ramified and $\mu(\sigma) \neq 0$ [**29**]. One can normalize the intertwining operators $A(\nu, \sigma, w)$ by a meromorphic (in ν) scalar factor to obtain a family of intertwining operators $\mathcal{A}(\nu, \sigma, w)$ with the following property [**16**,**26**]. If we let $\mathcal{A}(\sigma, w) = \mathcal{A}(0, \sigma, w)$, then these operators satisfy the cocycle condition

$$\mathcal{A}(\sigma, w_1 w_2) = \mathcal{A}(\widetilde{w}_2 \sigma, w_1) \mathcal{A}(\sigma, w_2),$$

for all $\widetilde{w}_1, \widetilde{w}_2 \in W(A)$. One consequence of this normalization is that the operators $\mathcal{A}(\nu, \sigma, w)$ are holomorphic on the unitary axis $i\mathfrak{a}^*$ [**29**]. Shahidi [**26**] has shown that the Plancherel measures and normalizing factors are related to conjectural Langlands L-functions.

Suppose $\widetilde{w}\sigma \simeq \sigma$. Choose an intertwining operator $T(w)$ with $T(w)(\widetilde{w}\sigma) = \sigma T(w)$. Then $\mathcal{A}'(\sigma, w) = T(w)\mathcal{A}(\sigma, w)$ is a self intertwining operator for $Ind_P^G(\sigma)$. Let $W(\sigma) = \{\widetilde{w} \in W(A) \mid \widetilde{w}\sigma \simeq \sigma\}$. Denote by $C(\sigma)$ the commuting algebra of $i_{G,M}(\sigma)$.

Theorem 1.2 (Harish–Chandra [**29**, Theorem 5.5.4.3]). *The collection $\{\mathcal{A}'(\sigma, w) \mid \widetilde{w} \in W(\sigma)\}$ spans the commuting algebra $C(\sigma)$.* $\square$

The theory of the Knapp–Stein R-group tells us how to determine a basis for $C(\sigma)$ from among the $\mathcal{A}'(\sigma, w)$. Let $\Phi(P, A)$ be the reduced roots of P with respect to A, and let $\beta \in \Phi(P, A)$. Let A_β be the torus $(\ker \beta \cap A)^o$. Let M_β denote the centralizer of A_β in G. Then M is a maximal proper Levi subgroup of M_β. Let $\mu_\beta(\sigma)$ be the Plancherel measure attached to $i_{M_\beta,M}(\sigma)$. Since M is a maximal proper Levi subgroup of M_β, we know $\mu_\beta(\sigma) = 0$ if and only if $\widetilde{w}\sigma \simeq \sigma$, for some $\widetilde{w} \neq 1$ in $W(M_\beta/A)$, and $i_{M_\beta,M}(\sigma)$ is irreducible. We denote by Δ' the collection of $\beta \in \Phi(P, A)$ such that $\mu_\beta(\sigma) = 0$. We let

$$R = R(\sigma) = \{\widetilde{w} \in W(\sigma) \mid \widetilde{w}\beta > 0, \ \forall \beta \in \Delta'\}.$$

Let W' be the subgroup of $W(\sigma)$ generated by the reflections in the roots of Δ'.

THEOREM 1.3 (KNAPP–STEIN, SILBERGER [**18**,**28**]). *For any* $\sigma \in \mathcal{E}_2(M)$, *we have* $W(\sigma) = R \ltimes W'$. *Furthermore,* $W' = \{\widetilde{w} \in W(\sigma) \mid \mathcal{A}'(\sigma, w)$ *is scalar* $\}$. $\square$

Thus, $\{\mathcal{A}'(w, \sigma) \mid \widetilde{w} \in R\}$ is a basis for $C(\sigma)$. The number of irreducible constituents of $i_{G,M}(\sigma)$ is the number of irreducible representations of R, and the representation corresponding to $\rho \in \hat{R}$ appears with multiplicity $\dim \rho$. Moreover, if $\widetilde{w}_1, \widetilde{w}_2 \in R$, then

$$\mathcal{A}'(\sigma, w, w_2) = \eta(w_1, w_2)\mathcal{A}'(\sigma, w_1)\mathcal{A}'(\sigma, w_2),$$

where the 2–cocycle $\eta: R \times R \to \mathbb{C}^*$ satisfies $T(w, w_2) = \eta(w_1, w_2)T(w_1)T(w_2)$. It is known that $C(\sigma) \simeq \mathbb{C}[R]_\eta$, where $\mathbb{C}[R]_\eta$ is the complex group algebra, twisted by the cocycle η. The multiplicity of each constituent of $i_{G,M}(\sigma)$ is equal to one if and only if R is abelian and η splits [**16**,**17**]. The isotypic components of $i_{G,M}(\sigma)$ can be parameterized by the irreducible representations of R [**17**].

We now assume that R is abelian and $C(\sigma) \simeq \mathbb{C}[R]$. For each $\widetilde{w} \in R$, we let $\mathfrak{a}_{\widetilde{w}} = \{H \in \mathfrak{a} \mid w \cdot H = H\}$. Let Z be the split component of G, and let $\mathfrak{z}$ be the real lie algebra of Z. Let $\mathfrak{a}_R = \bigcap\limits_{\widetilde{w} \in R} \mathfrak{a}_{\widetilde{w}}$.

THEOREM 1.4 (ARTHUR [**1**, PROPOSITION 2.1]).
Suppose R *is abelian and* $C(\sigma) \simeq \mathbb{C}[R]$. *Then the following are equivalent:*

(a) $i_{G,M}(\sigma)$ *has an elliptic constituent,*

(b) all the constituents of $i_{G,M}(\sigma)$ *are elliptic,*

(c) There is a $\widetilde{w} \in R$ *with* $\mathfrak{a}_{\widetilde{w}} = \mathfrak{z}$. $\square$

THEOREM 1.5 (HERB [**13**]). *Suppose* R *is abelian and* $C(\sigma) \simeq \mathbb{C}[R]$. *Let* π *be an irreducible constituent of* $i_{G,M}(\sigma)$. *Then* $\pi = i_{G,M'}(\tau)$ *for a proper Levi subgroup* M' *and some* $\tau \in \mathcal{E}_t(M')$ *if and only if* $\mathfrak{a}_R \neq \mathfrak{z}$. *Moreover,* M' *and* τ *can be chosen with* τ *elliptic if and only if there is a* $\widetilde{w}_0 \in R$ *with* $\mathfrak{a}_R = \mathfrak{a}_{\widetilde{w}_0}$. $\square$

We will use these last two theorems to describe the irreducible tempered representations of $SL_n(F)$ which are elliptic, and those which are not irreducibly induced from elliptic representations.

One of our main tools is the use of restriction theorems. We state those we need below. Tadic [**30**] has extended these results to the case where the quotient is not necessarily finite, but H is of the form $G_1Z(G)$, with G_1 the derived group of G.

THEOREM 1.6 (GELBART–KNAPP [**7**]). *Let G be a totally disconnected group, and suppose that H is an open normal subgroup of G, with G/H a finite abelian group.*

(a) *If π is an irreducible admissible representation of G, then $\pi|_H$ is the finite direct sum of irreducible admissible representations. Each component of $\pi|_H$ appears with the same multiplicity.*

(b) *If σ is an irreducible constituent of $\pi|_H$, and $G_\sigma = \{g \in G \mid g\cdot\sigma \simeq \sigma\}$, then G/G_σ permutes the inequivalent components of $\pi|_H$ simply and transitively. (Here $g\cdot\sigma(x) = \sigma(g^{-1}xg)$.)*

(c) *If σ is an irreducible admissible representation of H, then there is an irreducible admissible representation π_σ of G so that $\pi_\sigma|_H$ contains σ.*

(d) *Suppose π and π' are irreducible admissible representations of G such that both $\pi|_H$ and $\pi'|_H$ decompose with multiplicity one. Suppose σ is a constituent of both $\pi|_H$ and $\pi'|_H$. Then $\pi|_H \simeq \pi'|_H$, and $\pi' \simeq \pi\otimes\eta$, where η is a character of G, which is trivial on H.*

§2 The group SL_n .

Let F be as in Section 1. Let $\mathbf{G}_n = \mathbf{SL}_n$ and $\tilde{\mathbf{G}}_n = \mathbf{GL}_n$, as defined over F. We let $G_n = \mathbf{G}_n(F)$ and $\tilde{G}_n = \tilde{\mathbf{G}}_n(F)$. If the dimension is clear we may just write G or $\tilde{G}$. Let $\tilde{Z} = \tilde{Z}_n$ be the center of $\tilde{G}$.

Let $\tilde{A}_0 \subset \tilde{G}$ be the subgroup of diagonal matrices, and let $A_0 = G \cap \tilde{A}_0$. Let U be the subgroup of unipotent upper triangular matrices. Then $U \subset G$, and $\tilde{B} = \tilde{A}_0U$ is a Borel subgroup of $\tilde{G}$, while $B = A_0U$ is one of G. Let $\Phi(G,A_0) = \Phi(\tilde{G},\tilde{A}_0)$ be the roots of A_0 in G. Let $\Delta = \{e_i - e_{i+1}\}_{i=1}^{n-1}$ be the collection of simple roots given by B. Let $\theta \subset \Delta$, and let $\tilde{P}_\theta = \tilde{M}_\theta N_\theta$ be the associated standard parabolic subgroup of $\tilde{G}$. Then $P_\theta = \tilde{P}_\theta \cap G = M_\theta N_\theta$, with $M_\theta = \tilde{M}_\theta \cap G$, is a standard parabolic subgroup of G, and every standard parabolic arises in this way. Suppose $\tilde{M} = \tilde{M}_\theta$. Then there is a partition $m_1 + m_2 + \ldots + m_r = n$, such that

$$\tilde{M} \simeq \tilde{G}_{m_1} \times \tilde{G}_{m_2} \times \ldots \times \tilde{G}_{m_r}.$$

Specifically,

$$\tilde{M} = \left\{ \begin{pmatrix} g_1 & & & 0 \\ & g_2 & & \\ & & \ddots & \\ 0 & & & g_r \end{pmatrix} \;\middle|\; g_i \in \tilde{G}_{m_i} \right\}.$$

Then

$$M = \tilde{M} \cap G = \left\{ \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \;\middle|\; \begin{array}{l} g_i \in \tilde{G}_{m_i} \\ \det g_1 \cdot \det g_2 \cdot \ldots \cdot \det g_r = 1 \end{array} \right\}.$$

Let $\tilde{A} = \tilde{A}_\theta$ be the split component of $\tilde{M}$, and $A = \tilde{A} \cap G$ that of M. Then

$$\tilde{A} = \left\{ \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_r \end{pmatrix} \;\middle|\; \lambda_i \in F^\times \right\},$$

where by λ_i we really mean $\lambda_i I_{m_i}$. Thus,

$$A = \left\{ \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_r \end{pmatrix} \;\middle|\; \lambda_1^{m_1} \lambda_2^{m_2} \ldots \lambda_r^{m_r} = 1 \right\}.$$

The Weyl group $W = W(G/A) \simeq W(\tilde{G}/\tilde{A})$, is isomorphic to a subgroup of S_r. More precisely, W is generated by the transpositions (ij) for which $m_i = m_j$. If (ij) is in W, then

$$(ij) \cdot (\lambda_1, \ldots, \lambda_i, \ldots, \lambda_j, \ldots, \lambda_r) = (\lambda_1, \ldots, \lambda_j, \ldots, \lambda_i, \ldots, \lambda_r).$$

Let M_0 be the derived group of $\tilde{M}$.

$$M_0 = \left\{ \begin{pmatrix} g_1 & & \\ & \ddots & \\ & & g_r \end{pmatrix} \;\middle|\; g_i \in G_{m_i} \right\} \simeq G_{m_1} \times \ldots \times G_{m_r}.$$

Note that M_0 is also the derived group of M. Let $\varphi\colon M \longrightarrow \underbrace{F^\times \times \ldots \times F^\times}_{r-1 \text{ times}}$ be given by

$$\varphi(g_1, g_2, \ldots, g_r) = (\det g_1,\ \det g_2, \ldots, \det g_{r-1}).$$

We note that we have the following exact sequences.

$$(2.1) \quad 1 \to G_n\tilde{Z}_n \to \tilde{G}_n \overset{\det}{\to} F^\times/(F^\times)^n \to 1$$

$$(2.2) \quad 1 \to M\tilde{A} \to \tilde{M} \overset{\det}{\to} F^\times/\left((F^\times)^{m_1}(F^\times)^{m_2}\ldots(F^\times)^{m_r}\right) \to 1$$

$$(2.3) \quad 1 \to M_0A \to M \overset{\varphi}{\to} F^\times/(F^\times)^{m_1} \times F^\times/(F^\times)^{m_2} \times \ldots \times F^\times/(F^\times)^{m_{r-1}} \to 1.$$

We will choose specific splittings in order to simplify our later arguments. For each $m \geq 1$ let $\{a_{m,1}, a_{m,2}, \ldots a_{m,t_m}\}$ be a collection of representatives for $F^\times/(F^\times)^m$. For each (m,i), let $\overline{a}_{m,i} = \begin{pmatrix} a_{m,i} & \\ & I_{m-1} \end{pmatrix}$. Then $a_{n,i} \mapsto \overline{a}_{n,i}$ splits (2.1).

Similarly, if y is a representative for $F^\times/\left((F^\times)^{m_1}(F^\times)^{m_2}\ldots(F^\times)^{m_r},\right)$ then we let $\tilde{y} = \begin{pmatrix} y & \\ & I_{n-1} \end{pmatrix}$. Then $y \mapsto \tilde{y}$ splits (2.2). Now let

$$a = (a_{m_1,i_1}, a_{m_2,i_2}, \ldots a_{m_{r-1},i_{r-1}}) \in \prod_{j=1}^{r-1} F^\times/(F^\times)^{m_j}.$$

Let $\lambda(a) = a_{m_1,i_1} \cdot a_{m_2,i_2} \ldots a_{m_{r-1}i_{r-1}}$. Then we let

$$\psi(a) = \begin{pmatrix} \overline{a}_{m_1,i_1} & & & & \\ & \overline{a}_{m_2,i_2} & & & \\ & & \ddots & & \\ & & & \overline{a}_{m_{r-1}i_{r-1}} & \\ & & & & \overline{\lambda(a)}^{-1} \end{pmatrix}.$$

Clearly, ψ splits (2.3).

Note that if $\pi \in \mathcal{E}_2(\tilde{G}_n)$, and we write $\pi|_{G_n} = \bigoplus_j \rho_j$, then [**24**,**30**] each ρ_j appears with multiplicity one. Theorem 1.6(b) implies that the $\overline{a}_{n,i}$ permute the constituents ρ_j transitively. The representations ρ_j are said to form an L-packet for G_n. We also say that the ρ_j are L-indistinguishable.

Let $\sigma \in \mathcal{E}_2(M)$. Then, by Theorem 1.6(c), there is some $\pi_\sigma \in \mathcal{E}_2(\tilde{M})$ with $\pi_\sigma|_M \supset \sigma$. Moreover, if π'_σ is another such representation, then $\pi'_\sigma = \pi_\sigma \otimes \eta \cdot \det$, for some character η of $F^\times$ (Theorem 1.6(d)). We denote $\pi_\sigma \otimes \eta \cdot \det$ by $\pi_\sigma \otimes \eta$. Let $\pi_\sigma = \pi_1 \otimes \pi_2 \otimes \ldots \otimes \pi_r$, with each $\pi_i \in \mathcal{E}_2(\tilde{G}_{m_i})$. Let $\pi_\sigma|_M = \bigoplus_i \sigma_i$, with $\sigma_1 = \sigma$. We again say that the representations σ_i are L-indistinguishable, and say that $\{\sigma_i\}$ forms an L-packet of M. The reason for this terminology is discussed in [**7**]. If $w \in W(G/A)$, and we realize w as a permutation on r letters, then $w\pi_\sigma \simeq \pi_{w(1)} \otimes \pi_{w(2)} \otimes \ldots \otimes \pi_{w(r)}$.

Note that if $\pi_i|_{G_{m_i}} = \bigoplus_{j=1}^{b_i} \rho_{ij}$, then $\pi_\sigma|_{M_0} = \bigoplus_{\{j_i\}} \bigotimes_{i=1}^{r} \rho_{ij_i}$ is multiplicity free. Thus, for $i \neq k$, $\operatorname{Hom}_{M_0}(\sigma_k, \sigma_i) = \{0\}$. Note that this (redundantly) implies that $\pi_\sigma|_M$ is multiplicity free.

Lemma 2.1 (Shahidi [**24**]). *Let $\sigma \in \mathcal{E}_2(M)$ and choose $\pi_\sigma \in \mathcal{E}_2(\tilde{M})$ which contains σ upon restriction to M. Let $\alpha \in \Phi(P, A)$. Then*

(a) $i_{G,M}(\sigma) \hookrightarrow i_{\tilde{G},\tilde{M}}(\pi_\sigma)$;

(b) $i_{M_\alpha,M}(\sigma) \hookrightarrow i_{\tilde{M}_\alpha,\tilde{M}}(\pi_\sigma)$;

(c) $\mu_\alpha(\sigma) = \mu_\alpha(\pi_\sigma)$. □

For $1 \leq i \leq r$, let $c_i = \sum_{j=1}^{i} m_i$. For $1 \leq i < j \leq r$, let $\alpha_{ij} = e_{c_i} - e_{c_{j-1}+1}$. Then $\{\alpha_{ij} \mid 1 \leq i < j \leq r\}$ is a complete set of representatives for the reduced roots, $\Phi(P, A)$.

Corollary 2.2. *Let σ and π_σ be as in Lemma 2.1. Suppose $\pi_\sigma = \pi_1 \otimes \ldots \otimes \pi_r$. Then $\alpha_{ij} \in \Delta'$ if and only if $\pi_i \simeq \pi_j$.*

Proof. Let $\alpha = \alpha_{ij}$. Recall that $\alpha \in \Delta'$ if and only if $\mu_\alpha(\sigma) = 0$. By Lemma 2.1, $\mu_\alpha(\sigma) = 0$ if and only if $\mu_\alpha(\pi_\sigma) = 0$. By [**3**,**25**] this is equivalent to $\pi_i \simeq \pi_j$. □

We now describe the group $W(\sigma)$ in terms of the representation π_σ.

Lemma 2.3. *Let $\sigma \in \mathcal{E}_2(M)$, and suppose $\pi_\sigma \in \mathcal{E}_2(\tilde{M})$ with $\pi_\sigma|_M \supset \sigma$. Then*

$$W(\sigma) = \{w \in W \mid w\pi_\sigma \simeq \pi_\sigma \otimes \eta, \text{ for some } \eta \in \hat{F^\times}\}.$$

Remark. That $w\sigma \simeq \sigma$ implies $w\pi_\sigma \simeq \pi_\sigma \otimes \eta$ for some η was proved by Shahidi in [**24**].

Proof. If $w\sigma \simeq \sigma$, then $w\sigma \hookrightarrow \pi_\sigma|_M$. Since $w\sigma \subset w\pi_\sigma|_M$, we know that $\pi_\sigma|_M$ and $w\pi_\sigma|_M$ have a common constituent. Thus, since $\pi_\sigma|_M$ and $w\pi_\sigma|_M$ are multiplicity free, Theorem 1.6(d) implies that $w\pi_\sigma \simeq \pi_\sigma \otimes \eta$, for some $\eta \in \hat{F^\times}$.

Now suppose that $w\pi_\sigma \simeq \pi_\sigma \otimes \eta$. Then we know that $w\sigma \simeq \sigma_i$ for some i. Note that $w\pi_\sigma|_{M_0} = \bigoplus_{\{j_i\}} \bigotimes_{i=1}^{r} \rho_{w(i)j_{w(i)}}$. Suppose $\rho_0 = \bigotimes_{i=1}^{r} \rho_{ij_i}$ is an irreducible constituent of $\sigma|_{M_0}$. Since $w\pi_\sigma \simeq \pi_\sigma \otimes \eta$, we know that $\pi_{w(i)} \simeq \pi_i \otimes \eta$ for each i. Thus, $\rho_{w(i)j_{w(i)}}$ and ρ_{ij_i} are L-indistinguishable. By Theorem 1.6(b) there is a choice of k_i so that $\overline{a}_{m_i,k_i} \cdot \rho_{ij_i} = \rho_{w(i)j_{w(i)}}$. Suppose $s = (i\ w(i)\ w^2(i) \ldots w^{\ell-1}(i))$ is a cycle appearing in w. Without loss of generality, assume $s = (1\ 2\ \ldots\ \ell)$. Let m be the common value of $m_1, m_2, \ldots, m_\ell$. For each $1 \le i \le \ell - 1$, we choose $b_i = a_{m,k_i}$ with the property that $\overline{b}_i \cdot \rho_{ij_i} = \rho_{(i+1)j_{i+1}}$. Let $b_\ell = (b_1 b_2 \ldots b_{\ell-1})^{-1}$. Then, since the $\overline{b}_i$ commute,

$$\overline{b}_\ell \cdot \rho_\ell = (\overline{b}_1 \ldots \overline{b}_{\ell-1})^{-1} \rho_{\ell j_\ell} = \rho_{1j_1}.$$

That is, we can take $a_{m,k_\ell} = b_\ell$. Therefore, we can choose a_{m_i,k_i} so that their product over any cycle s of w is 1, and thus the product of all a_{m_i,k_i} is 1.

Let

$$\overline{b} = \begin{pmatrix} \overline{a}_{m_1,k_1} & & & & & \\ & \overline{a}_{m_2,k_2} & & & & \\ & & \ddots & & & \\ & & & \ddots & & \\ & & & & \ddots & \\ & & & & & \overline{a}_{m_r,k_r} \end{pmatrix}.$$

Then we have just shown that $\overline{b} \in M$. Thus, by Theorem 1.6(b), $\overline{b} \cdot \rho_0$ is a constituent of $\sigma|_{M_0}$. On the other hand,

$$\overline{b} \cdot \rho_0 = \bigotimes_{i=1}^{r} \overline{a}_{m_i,k_i} \cdot \rho_{ij_i} = \bigotimes_{i=1}^{r} \rho_{w(i)j_{w(i)}} = w\rho_0.$$

Thus, $w\rho_0 \subset \sigma$ and $w\rho_0 \subset w\sigma$ implies $\mathrm{Hom}_{M_0}(\sigma, w\sigma) \neq \{0\}$. Therefore, by multiplicity one, $\sigma \simeq w\sigma$. $\square$

Let

$$\overline{L}(\pi_\sigma) = \{\eta \in \widehat{F}^\times \mid \pi_\sigma \otimes \eta \simeq w\pi_\sigma, \text{ for some } w \in W\}.$$

Let $X(\pi_\sigma) = \{\eta \in F^\times \mid \pi_\sigma \otimes \eta \simeq \pi_\sigma\}$. Note that if $\eta, \chi \in \overline{L}(\pi_\sigma)$, and $\pi_\sigma \otimes \eta \simeq \pi_\sigma \otimes \chi$, then $\eta\chi^{-1} \in X(\pi_\sigma)$. Thus, there is a well defined homomorphism $\varphi: W(\sigma) \to \overline{L}(\pi_\sigma)/X(\pi_\sigma)$ given by $\varphi(w) = \eta X(\pi_\sigma)$, where $w\pi_\sigma \simeq \pi_\sigma \otimes \eta$.

THEOREM 2.4. *The R-group of σ is given by*

$$R(\sigma) \simeq \overline{L}(\pi_\sigma)/X(\pi_\sigma).$$

Proof. It is enough to show that $\ker\varphi = W'$, where W' is the group generated by reflections in the roots of Δ'. If $\alpha_{ij} \in \Delta'$, then $\pi_i \simeq \pi_j$, so $(ij) \cdot \pi_\sigma \simeq \pi_\sigma$, and thus, $W' \subseteq \ker\varphi$. On the other hand suppose $w = s_1 s_2 \dots s_k$ is in $\ker\varphi$. Let $s_i = (i_1\ i_2\ \dots\ i_j)$. Since $w\pi_\sigma \simeq \pi_\sigma$, $\pi_{i_\ell} \simeq \pi_{i_{\ell+1}}$ for $1 \le \ell \le j-1$. Thus, by Corollary 2.2, $\alpha_{i_\ell i_{\ell+1}} \in \Delta'$, for each ℓ. Let $\alpha_{i_j i_{j+1}} = \alpha_{i_1 i_j}$. Then

$$w = \prod_{i=1}^{k} \prod_{\ell=1}^{i_j} w_{\alpha_{i_\ell i_{\ell+1}}} \in W'.$$

Thus, $\ker\varphi = W'$, so $\overline{L}(\pi_\sigma)/X(\pi_\sigma) \simeq W(\sigma)/W' \simeq R$. ∎

Remark. The fact that $W' = \{w \mid w\pi_\sigma \simeq \pi_\sigma\}$ was first shown, with a slightly different proof, by Shahidi [**24**, Proposition 1.8].

Remark. If P is the minimal parabolic, then Gelbart and Knapp [**7**] showed that $\overline{L}(\pi_\sigma) \simeq R(\sigma)$. Thus, our result generalizes theirs, as well as those of Keys [**16**].

COROLLARY 2.5. *If σ and σ' are L-indistinguishable discrete series representations of M, then $R(\sigma) = R(\sigma')$.* ∎

While Theorem 2.4 describes R as a subgroup of $(F^\times/(F^\times)^n)^\wedge$, we desire a more explicit description of R. Let $\eta \in \overline{L}(\pi_\sigma)$. Let $\Omega(\eta, i) = \{j \mid \pi_j \simeq \pi_i \otimes \eta\}$. Let $w_\eta(1) = \min\Omega(\eta, 1)$. For $2 \le i \le r$, let $\Gamma(\eta, i) = \{w_\eta(j) \mid j < i\}$. Then we let

$$w_\eta(i) = \min\left(\Omega\left(\eta, i\right) \cap \left(\Gamma\left(\eta, i\right)\right)^c\right).$$

Clearly $w_\eta \in W$.

THEOREM 2.6. *Let* $\eta \in \overline{L}(\pi_\sigma)$. *Then* w_η *is the unique element of* $R(\sigma)$ *associated with* η.

Proof. Since, for each i, $\pi_{w_\eta(i)} \simeq \pi_i \otimes \eta$, we have $w_\eta \pi_\sigma \simeq \pi_\sigma \otimes \eta$. Thus, $w_\eta \in W(\sigma)$. Suppose $\alpha_{ij} \in \Delta'$. Then $\pi_i \simeq \pi_j$, so $\Omega(\eta, i) = \Omega(\eta, j)$. Since $i < j$, we have $w_\eta(i) < w_\eta(j)$, by construction. Thus, $w_\eta \alpha_{ij} = \alpha_{w_\eta(i) w_\eta(j)} > 0$. Therefore, for each $\alpha \in \Delta', w_\eta \alpha > 0$, and thus $w_\eta \in R(\sigma)$. □

§3 Elliptic representations.

We now use our description of the R-groups of G to explicitly describe the elliptic tempered spectrum of G. We also describe those tempered representations which are not elliptic, and are not irreducibly induced from an elliptic representation. We begin with the multiplicity one result of Howe and Silberger. This result has been extended to an arbitrary irreducible admissible unitary representation of M [**30**].

THEOREM 3.1 (HOWE–SILBERGER [**14**]). *Let* $G = SL_n(F)$, *and let* $P = MN$ *be an arbitrary parabolic subgroup of* G. *Suppose* $\sigma \in \mathcal{E}_2(M)$. *Then each constituent of* $i_{G,M}(\sigma)$ *appears with multiplicity one.* □

COROLLARY 3.2. *For any* $\sigma \in \mathcal{E}_2(M)$, $C(\sigma) \simeq \mathbb{C}[R]$. □

LEMMA 3.3. *Let* $P = MN$ *be a standard parabolic subgroup of* G. *Let* $\tilde{M}$ *be the Levi subgroup of* $\tilde{G}$ *with* $M = \tilde{M} \cap G$. *Suppose* $\tilde{M} \simeq \tilde{G}_{m_1} \times \tilde{G}_{m_2} \times \ldots \times \tilde{G}_{m_r}$. *If, for some* i *and* $j, m_i \neq m_j$, *then* $i_{G,M}(\sigma)$ *can never contain an elliptic constituent.*

Proof. By Theorem 1.4 and Corollary 3.2, $i_{G,M}(\sigma)$ has an elliptic constituent if and only if there is a $w \in R$ so that $\mathfrak{a}_w = \mathfrak{z}$. Since $m_i \neq m_j$, $W(G/A)$ does not permute the blocks of M transitively. Thus, there is no $w \in W(G/A)$ with $\mathfrak{a}_w = \mathfrak{z} = \{0\}$. Therefore, for any $\sigma \in \mathcal{E}_2(M)$, $i_{G,M}(\sigma)$ cannot contain an elliptic constituent. □

THEOREM 3.4. *Suppose* $m_1 = m_2 = \ldots = m_r$. *Let* $\sigma \in \mathcal{E}_2(M)$, *and choose* $\pi_\sigma \in \mathcal{E}_2(\tilde{M})$ *with* $\pi_\sigma|_M \supset \sigma$. *Then the following are equivalent:*

(a) $i_{G,M}(\sigma)$ *has an elliptic constituent,*

(b) *every constituent of* $i_{G,M}(\sigma)$ *is elliptic,*

(c) $R(\sigma) \simeq \mathbb{Z}_r$.

Proof. Since R is abelian and $C(\sigma) \simeq \mathbb{C}[R]$, (1) and (2) are equivalent, and both are equivalent to $\mathfrak{a}_w = \{0\}$ for some $w \in R(\sigma)$. Since $m_1 = \ldots = m_r$, $W(G/A) \cong S_r$, and $\mathfrak{a}_w = \{0\}$ if and only if w is an n-cycle. Up to conjugation by an element of $W(G/A_0)$, we can assume that $w = (12\ldots r)$. Let $\pi_\sigma = \pi_1 \otimes \cdots \otimes \pi_r$, with each $\pi_i \in \mathcal{E}_2(\tilde{G}_m)$. From Theorem 2.6, $w \in R(\sigma)$ if and only if there is an $\eta \in \hat{F^\times}$ such that $\eta^r \in X(\pi_1)$, and $\eta^j \notin X(\pi_1)$ for $1 \leq j \leq r-1$, with $\pi_i = \pi_1 \otimes \eta^{i-1}$. That is,

$$\pi_\sigma \simeq \pi_1 \otimes (\pi_1 \otimes \eta) \otimes (\pi_1 \otimes \eta^2) \otimes \ldots \otimes (\pi_1 \otimes \eta^{r-1}).$$

Now it is clear that $\overline{L}(\pi_\sigma)/X(\pi_\sigma) = <\eta>$, so $R(\sigma) \simeq \mathbb{Z}_r$. $\square$

Remark. It is not the case that every irreducible tempered representation of G is either elliptic, or is irreducibly induced from an elliptic representation. This was already known for $G = SL_4$, with $P = B$, the Borel subgroup [**13**]. We will give a description of all representations of G of this form. We begin with an example which illustrates the ideas involved. This example is a generalization of the example given in [**13**] for SL_4.

Example 3.5. Let $m \geq 1$, and let $G = SL_{4m}$. Let $\tilde{M} \simeq \tilde{G}_m \times \tilde{G}_m \times \tilde{G}_m \times \tilde{G}_m$. Let $\pi \in \mathcal{E}_2(M)$. Suppose that η and χ are distinct characters with η, χ and $\eta\chi \notin X(\pi)$, but $\eta^2, \chi^2 \in X(\pi)$. Let

$$\pi_0 = \pi \otimes (\pi \otimes \eta) \otimes (\pi \otimes \chi) \otimes (\pi \otimes \eta\chi).$$

Let $\sigma \subset \pi_0|_M$. Then $\Delta' = \emptyset$. Note that η corresponds to the permutation (12)(34), χ to (13)(24), and $\eta\chi$ to (14)(32). These are the non–trivial elements of $R(\sigma)$. Note that $\mathfrak{a}_R = \{0\}$, but for each $w \in R(\sigma)$, $\mathfrak{a}_w \supsetneq \{0\}$. Therefore, by Theorem 1.5, no constituent of $i_{G,M}(\sigma)$ is irreducibly induced from an elliptic representation. $\square$

Definition 3.6. Let $\pi \in \mathcal{E}_2(\tilde{G}_m)$. Let $\eta_1, \eta_2, \ldots, \eta_\ell, \ell \geq 2$, be a collection of characters of $F^\times$. Let $o(\eta_i)$ be the order of η_i modulo $X(\pi)$. Suppose that

(1) $\eta_1^{i_1}\eta_2^{i_2}\ldots\eta_\ell^{i_\ell} \notin X(\pi)$ unless $\eta_j^{i_j} \in X(\pi)$ for each j ;

(2) $gcd(o(\eta_i))_{i=1}^{\ell} > 1$.

Let $\Omega(\pi,\eta_1,\eta_2,\ldots,\eta_\ell) = \left\{ \pi \otimes \eta_1^{i_1}\eta_2^{i_2}\ldots\eta_\ell^{i_\ell} \;\middle|\; \begin{matrix} 0 \le i < o(\eta_j) \\ \\ j = 1,\ldots \ell \end{matrix} \right\}$. We call the collection $\Omega(\pi,\eta_1,\eta_2,\ldots,\eta_\ell)$ a <u>multiple character segment</u> for π.

Definition 3.7. Let $\tilde{G} = \tilde{G}_n$. Suppose $\tilde{P} = \tilde{M}N$ is a standard parabolic of $\tilde{G}$. A discrete series representation ρ of $\tilde{M}$ is said to <u>contain</u> a multiple character segment, Ω for π if, up to permutation of the blocks of M,

$$\rho \cong (\bigotimes_{\tau\in\Omega} \tau) \otimes \rho',$$

for some ρ'.

THEOREM 3.8. *Let $\sigma \in \mathcal{E}_2(M)$, and choose $\pi_\sigma \in \mathcal{E}_2(\tilde{M})$ with $\pi_\sigma|_M \supset \sigma$. Then any constituent of $i_{G,M}(\sigma)$ is non–elliptic, and is not irreducibly induced from an elliptic representation if and only if π_σ contains a multiple character segment $\Omega(\pi,\eta_1,\ldots,\eta_\ell)$, with each $\eta_i \in \overline{L}(\pi_\sigma)$.*

Proof. Suppose $\pi_\sigma \simeq \pi_1\otimes\ldots\otimes\pi_r$, and $\{\pi_1,\ldots,\pi_k\}$ is a multiple character segment $\Omega(\pi_1,\eta_1,\ldots,\eta_\ell)$. Further suppose that $\eta_i \in \overline{L}(\pi_\sigma)$ for each η_i. Then $w_{\eta_i} \neq 1$, since for $1 \le j \le k$, $\pi_j \otimes \eta_i \not\simeq \pi_j$. For $1 \le j \le k$ there are $i_1, i_2, \ldots, i_\ell$ so that

$$\pi_1 \otimes \eta_1^{i_1}\eta_2^{i_2}\ldots\eta_\ell^{i_\ell} \cong \pi_j.$$

Thus, there is a $w \in R(\sigma)$, with $w(1) = j$, for $j = 1,2,\ldots,k$. Let m denote the common value of $m_1,\ldots,m_k$. Then,

$$\mathfrak{a}_R \subseteq \left\{ \begin{pmatrix} d &&&&&& \\ & d &&&&& \\ && \ddots &&&& \\ &&& d &&& \\ &&&& d_{k+1} && \\ &&&&& \ddots & \\ &&&&&& d_r \end{pmatrix} \;\middle|\; mdk + \sum_{k+1}^{r} d_i m_i = 0 \right\}.$$

We denote the subalgebra on the right by $\mathfrak{a}'$. Since $gcd(o(\eta_i)) \geq 2$, there is no character η so that, for each $2 \leq j \leq k$, $w_\eta^t(1) = j$ for some t. Thus, there is no $w \in R$ with $\mathfrak{a}_w \subset \mathfrak{a}'$, and thus it is impossible for $\mathfrak{a}_w = \mathfrak{a}_R$ for some $w \in R$. Therefore, by Theorem 1.5, every component of $i_{G,M}(\sigma)$ is non–elliptic, and cannot be irreducibly induced from an elliptic representation.

Now suppose that π_σ does not contain a multiple character segment with the described compatibility condition. Suppose that $w(i) \neq i$ for some $w \in R$. Since there is no compatible multiple character segment, we know there is a character, $\gamma_i = \eta_k$ for some k, so that $\pi_{w(i)} = \pi_i \otimes \gamma_i^j$ for some j. That is, we choose $\gamma_i \in \overline{L}(X(\pi_\sigma))$ so that the order of γ_i modulo $X(\pi_\sigma)$ is maximal, with the property that $\pi_i \otimes \gamma_i \not\simeq \pi_i$. Let $s(i)$ be the cycle of w_{γ_i} which contains i. Note that if $w \in R$, and $w(i) \neq i$, then some power of $s(i)$ appears in W. (This follows from the construction of the elements w_η of R.) Suppose that, $\gamma_k \not\equiv \gamma_i^j$ mod $(X(\pi_\sigma)$ for any $1 \leq j \leq o(\gamma_i) - 1$. Then $w_{\gamma_k}(i) = i$, and so $\pi_i \otimes \gamma_k \simeq \pi_i$. Let $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_\ell}$ be the distinct classes, modulo $X(\pi_\sigma)$, among the characters $\{\gamma_j\}$. Let $w_0 = w_{\gamma_{i_1}} w_{\gamma_{i_2}} \ldots w_{\gamma_{i_\ell}}$. By construction, the elements $w_{\gamma_{i_j}}$ are disjoint permutations, and $w_0 \in R$. Moreover, if there is a $w \in R$ with $w(i) = k$, then $w_0^j(i) = k$ for some j. Thus, $\mathfrak{a}_{w_0} = \mathfrak{a}_R$. Therefore, by Theorem 1.5, if $i_{G,M}(\sigma)$ has no elliptic constituents, then each constituent of $i_{G,M}(\sigma)$ can be irreducibly induced from an elliptic representation of some proper Levi subgroup M' of G. $\square$

Remark. Suppose $\sigma \in \mathcal{E}_2(M)$ and all the constituents of $\pi = i_{G,M}(\sigma)$ are elliptic. We can parameterize the constituents by the characters $\hat{R}$ of R. Let π_κ be the constituent which corresponds to $\kappa \in \hat{R}$. Then $\Theta_\pi^e = 0$, so $\sum\limits_\kappa \Theta_{\pi_\kappa}^e = 0$. We would like to explicitly know this relation between the characters $\Theta_{\pi_\kappa}^e$. In [**13**] Herb gives an explicit description of this character relation when $G = Sp_{2n}$ or SO_n. In [**10**] we used the same techniques to carry out this program when $G = U_n$. Assem [**2**] uses his global character expansions, and a result of Kazhdan [**15**] to describe this relation when $G = G_n$, and n is prime. Shahidi [**24**] showed that $R(\sigma) \simeq X(i_{\tilde{G},\tilde{M}}(\pi_\sigma))/X(\pi_\sigma)$. Thus, $\overline{L}(\pi_\sigma) = X(i_{\tilde{G},\tilde{M}}(\pi_\sigma))$. Therefore, by extending the results of Kazhdan, one hopes to describe this relation.

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