PACIFIC JOURNAL OF MATHEMATICS
Vol. 46, No. 2, 1973

THE TOTAL SPACE OF UNIVERSAL FIBRATIONS

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It is shown that the total space of a universal fibration for a fibre F is a classifying space for the monoid of self homotopy equivalences of F which fix the base point.

For any space F, there exists a universal Hurewicz fibration $F \to E_\infty \to B_\infty$, where B_∞ is a CW complex which classifies Hurewicz fibrations over CW complexes (See Dold [2], Theorem 16.9.). Now B_∞ is the Dold-Lashof classifying space for the monoid of self homotopy equivalence of F, which we shall denote by F^F. At least, this is the case when F is a CW complex. (See Allaud [1], § IV.) The purpose of this note is to show that E_∞ is the classifying space for the monoid of self equivalences of F leaving the base point fixed, denoted F_0^F, when F is a CW complex with homotopy equivalent path components. In fact, we shall show E_∞ is the base space of a Serre fibration with fibre F_0^F and a total space which is essentially contractible. We need this characterization of E_∞ in order to calculate the induced homomorphism on integral cohomology of the evaluation map $\omega: X^X \to X$ where $X = CP^n$. This is done in [3].

Let $D = p^*(E_\infty)$, the pullback of E_∞ by $p: E_\infty \to B_\infty$. Thus $D = \{(e, e') \in E_\infty \times E_\infty \mid p(e) = p(e')\}$, and $\bar{p}: D \to E_\infty$ given by $\bar{p}(e, e') = e$ is the projection. Let D_0^F be the set of maps of $F \to D$ endowed with the C-0 topology such that:

(a) Each map carries F into some fibre of $D \xrightarrow{\bar{p}} E_\infty$ and is a homotopy equivalence of F and the fibre.

(b) Each map carries the base point, $*$, into a point of the form (e, e).

Let $q: D_0^F \to E_\infty$ be given by $q(f) = \bar{p} \circ f(*)$.

THEOREM.

(1) *q is a Serre Fibration, and if F is locally compact q is a Hurewicz fibration.*

(2) *The fibre of q is F_0^F.*

(3) *There is a fibrewise action $D_0^F \times F_0^F \to D_0^F$ if F is locally compact.*

(4) *D_0^F is essentially contractible.*

Proof of (1). First note that q is onto since all the components of F have the same homotopy type.

We shall assume that F has a whisker. That is, assume F has the form $J \vee F'$ where J is the unit interval with the base point $*$ being the 1 and with $0 \in F'$. Every space is homotopy equivalent to a space of this type, so we do not lose any generality.

Let X be a compact polyhedron (or F is locally compact). We must show that p has the covering homotopy property with respect to any map $X \to D_0^F$. Since X is compact (or F is locally compact) $X \times F$ is a CW complex; so we may consider the adjoint map

$$f: X \times F \longrightarrow D ,$$

where f is a fibre preserving map which carries $X \times *$ into $\varDelta \subset D$, where $\varDelta = \{(e, e) \mid e \in E_\infty\}$. Then the covering homotopy property translates into a statement involving a fibre homotopy of f which at each stage sends $X \times *$ into $\varDelta$. Reflecting on the definition of D, we see that the covering homotopy property is equivalent to the following statement: Let $f: X \times F \to E_\infty$ be a fibre map and let $h_t: X \to E_\infty$ be any homotopy such that $h_0(x) = f(x, *)$ for all $x \in X$. Then there exists a fibre homotopy $\tilde{h}_t: X \times F \to E_\infty$ such that $\tilde{h}_t(x, *) = h_t(x)$ and $\tilde{h}_0 = f$.

Now this statement is a special case of the statement that q has the covering homotopy extension property for the space $X \times F$ relative to $X \times *$. See Hu, page 62 [4] for the definition. But this follows from Satz 5.38, page 107 in [5]. (The fact that $*$ is on the end of a whisker allows us to satisfy the technical requirements of Satz 5.38 concerning a halo about $X \times *$.)

Proof of (4). First note that (2) and (3) are obviously true. The action in (3), $D_0^F \times F_0^F \to D_0^F$ is given by $(g, f) \to g \circ f$. (This action is continuous if F is locally compact.)

Now we shall show that D_0^F is essentially contractible. That is, any map $X \to D_0^F$ is homotopy trivial if X is a finite CW complex. Consider the adjoint map $g: X \times F \to D$. Then g is defined by, and defines, a fibre map $h: X \times F \to E_\infty$ by means of the relation

$$g(x, y) = (h(x, *), h(x, y)) \in D .$$

Now h can be extended to a fibre map $H: CX \times F \to E_\infty$, (because E_∞^F is essentially contractible, [1] Theorem 4.1). We define a fibre map $G: CX \times F \to D$ by

$$G(x, y) = (H(x, *), H(x, y)) , \qquad x \in CX, y \in F .$$

Note that G extends g. The adjoint situation now shows that our original map $X \to D_0^F$ factors through $X \to CX \to D_0^F$.

COROLLARY.

$$\pi_i(E_\infty) \cong \pi_{i-1}(F_0^F) .$$

This corollary plays an important role in the computation of the homomorphisms induced in cohomology by the evaluation map $\omega: F^F \to F$. The program is based on the use of the Federer spectral sequence and obstruction theory to compute some homotopy groups of F_0^F, and hence of E_∞ by the corollary. From the homotopy group information, we obtain information about the cohomology of E_∞. Then we use the Serre exact sequence and the slogan "ω^* factors through the transgression" to recover information about ω^*.

This program has been used successfully to compute ω^* for $H^2(CP^n; Z)$. See Theorem 16 of [3].

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Received March 27, 1972 and in revised form June 21, 1972. This research was partially supported by a grant from the National Science Foundation.

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