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FIBRE BUNDLES AND THE EULER CHARACTERISTIC

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1. Introduction

For any fibre bundle $F \xrightarrow{i} E \xrightarrow{p} B$ there are three important maps: the projection p, the fibre inclusion i, and the evaluation $\omega : \Omega B \to F$. In this paper we demonstrate formulas for each of these maps involving the Euler-Poincaré number of the fibre.

Let M be a compact topological manifold with possibly empty boundary $\dot{M}$, $\chi(M)$ the Euler-Poincaré number of M, G any space of homeomorphisms of M with a continuous action on M, $\omega : G \to M$ the evaluation map for some base point, $M \xrightarrow{i} E \xrightarrow{p} B$ any (locally trivial) fibre bundle, and $L \subset B$ a (possibly empty) subcomplex of the CW complex B.

Theorem A. *For connected M and any coefficients*

$$\chi(M)\omega^* = 0 : \tilde{H}^*(M) \to \tilde{H}^*(G) .$$

Theorem B. *There exists a transfer homomorphism $\tau : H^*(E, p^{-1}(L)) \to H^*(B, L)$ such that $\tau \circ p^* = \chi(M)1$ for any coefficients.*

Theorem C. *There exists a transfer homomorphism $\tau : H_*(B, L) \to H_*(E, p^{-1}(L))$ such that $p_* \circ \tau = \chi(M)1$ for any coefficients.*

Special cases of Theorem A were discovered by the author in [3] and [4]. Note that B and C reduce to the classical transfer theorem for covering spaces when M is a finite set of points. Borel proved a version of Theorem B for M a closed connected differentiable manifold and $M \xrightarrow{i} E \xrightarrow{p} B$ an "oriented" fibre bundle with structural group acting differentially on M and cohomology groups with fields of coefficients whose characteristics does not divide $\chi(M)$, [2]. This result was improved by the author in [1] and [3].

All these theorems are consequences of the next. Let $\dot{E}$ be the subspace of E consisting of those points of E which are in the boundaries of the fibres containing them. Then $(M, \dot{M}) \xrightarrow{i} (E, \dot{E}) \xrightarrow{p} B$ is a fibre pair. If $\dot{M}$ is empty, then $\dot{E}$ is empty.

Theorem D. *Let M^n be orientable and connected, and assume $\pi_1(B)$ acts*

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trivially on $H^n(M^n, \dot{M}; Z) \cong Z$. *Then there exists a* $\chi \in H^n(E, \dot{E}; Z)$ *such that* $i^*(\chi) = \chi(M)\mu$ *where* μ *generates* $H^n(M, \dot{M}; Z)$.

The author would like to acknowledge several conversations with J. C. Becker which greatly helped him on several occasions.

2. Integration along the fibre

Here we record some well known facts concerning integration along the fibre. Suppose $(F, F') \to (E, E') \xrightarrow{p} B$ is a fibred pair, and L is a subcomplex of B. Then the Serre spectral sequence converges to $H^*(E, E' \cup p^{-1}(L); G))$ and $E_2^{p,q} \cong H^p(B, L; \{H^q(F, F'; G)\})$.

Suppose $\pi_1(B)$ operates trivially on $H^n(F, F'; Z) \cong Z$ and $H^i(F, F'; Z) \cong 0$ for $i > n$. Then integration along the fibre is defined as the composition

$$p_\natural : H^i(E, E' \cup p^{-1}(L)) \twoheadrightarrow E_\infty^{i-n,n} \rightarrowtail E_2^{i-n,n} \cong H^{i-n}(B, L; H^n(M, M'; G) \cong H^{i-n}(B, L; G) .$$

Integration along the fibre satisfies three properties:

a) If $E \xrightarrow{p} E' \xrightarrow{q} B$ are two fibrations, then

$$(q \circ p)_\natural = q_\natural \circ p_\natural .$$

b) If we have a fibre square

$$\begin{tikzcd}
(F, F') \arrow[r] \arrow[d] & (\bar{F}, \bar{F}') \arrow[d] \\
(E, E' \cup p^{-1}(L)) \arrow[r, "\bar{f}"] \arrow[d, "p"] & (\bar{E}, \bar{E}' \cup \bar{p}^{-1}(\bar{L})) \arrow[d, "\bar{p}"] \\
(B, L) \arrow[r, "f"] & (\bar{B}, \bar{L})
\end{tikzcd}$$

and (F, F') and $(\bar{F}, \bar{F}')$ both have cohomological dimension n, then

$$\begin{tikzcd}
H^i(E, E' \cup p^{-1}(L)) \arrow[d, "p_\natural"] & H^i(\bar{E}, \bar{E} \cup p^{-1}(\bar{L})) \arrow[l, "\bar{f}^*"'] \arrow[d, "\bar{p}_\natural"] \\
H^{i-n}(B, L; G) & H^{i-n}(\bar{B}, \bar{L}; G) \arrow[l, "\psi"']
\end{tikzcd}$$

commutes, where ψ is induced by f^* and a homomorphism on the coefficient group corresponding to the map induced by $\bar{f} \,|\, (F, F')$.

c) If $u \in H^*(B, L; G)$ and $v \in H^*(E, E'; G')$ then $p_\natural(p^*(u) \cup v) = u \cup p'_\natural(v) \in H^*(B, L; G'')$, where G and G' pair to G'' and $p_\natural : H^*(E, E' \cup p^{-1}(L)) \to H^*(B, L)$, and $p'_\natural : H^*(E, E') \to H^*(B)$.

Dually, we may define $p^\natural$ as the composition

$$H_{i-n}(B, L; G) \cong E^2_{i-n,n} \twoheadrightarrow E^\infty_{i-n,n} \rightarrowtail H_i(E, E' \cup p^{-1}(L); G) .$$

Properties a) and b) hold in a dual formulation. For cap products

$$\cap : H^q(X, A_1; G) \otimes H_n(X, A_1 \cup A_2; G') \to H_{n-q}(X, A_2; G'')$$

we have the following formula:

$$p_*(a \cap p^\natural(y)) = p_\natural(\alpha) \cap y \in H_*(B, L; G'') ,$$

where $y \in H_*(B, L; G')$, $\alpha \in H^*(E, E'; G)$, $p^\natural : H_*(B, L) \to H_*(E, E' \cup p^{-1}(L))$, and $p_\natural : H^*(E, E') \to H^*(B)$.

3. Proof of Theorem D

Let G be a group of orientation-preserving homeomorphisms on M with compact-open topology acting transitively on $\mathring{M} = M - \dot{M}$. Let H be the subgroup of G leaving the base point $*$ fixed. We take $* \in \mathring{M}$.

Consider the universal principal bundle $G \to E_G \to B_G$. Then the classifying space for H is $B_H = E_G \times_G \mathring{M}$ since $G/H = \mathring{M}$. Let $\bar{B}_H$ denote $E_G \times_G M$, and let $\dot{B}_H$ denote $E_G \times_G \dot{M}$. We have the following diagram of fibre squares:

$$(1) \quad \begin{tikzcd}
M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M \arrow[d] \\
\pi^*(\bar{B}_H) \arrow[r, "\tilde{j}"] \arrow[d, "p"] & \bar{\pi}^*(\bar{B}_H) \arrow[r] \arrow[d, "\bar{p}"] & \bar{B}_H \arrow[d, "\bar{\pi}"] \\
B_H \arrow[r, "j"] \arrow[rr, bend right, "\pi"'] & \bar{B}_H \arrow[r, "\bar{\pi}"] & B_G
\end{tikzcd}$$

Here j and $\tilde{j}$ are inclusion maps.

Lemma 1. *Regarding* $\tilde{j}$ *as a map of pairs*

$$\tilde{j} : (\pi^*(\bar{B}_H), \pi^*(\dot{B}_H)) \to (\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) .$$

Then j *and* $\tilde{j}$ *are homotopy equivalences.*

Lemma 2. $(\pi^*(\bar{B}_H), \pi^*(\dot{B}_H)) = (E_G \times_H M, E_G \times_H \dot{M})$.

Proof.

$$\begin{tikzcd}
M \arrow[d] & & M \arrow[d] \\
E_G \times_H M \arrow[rr] \arrow[d] & & E_G \times_G M = \bar{B}_H \arrow[d] \\
E_G \times_G \mathring{M} = E_G/H = B_H \arrow[rr, "\pi"] & & B_G = E_G/G .
\end{tikzcd}$$

The existence of this fibre square implies that $E_G \times_H M = \bar{\pi}^*(\bar{B}_H)$.

Since M is oriented, $Z \cong H^n(M, M - *) \xrightarrow{i^*} H^n(M, \dot{M})$ is an isomorphism where i is inclusion. Thus by Lemmas 1 and 2 and the naturality of integration along the fibre (§ 2(b)) we have the following commutative diagram:

$$
(2)\qquad
\begin{tikzcd}
H^n(E_G \times_H M, E_G \times_H (M - *)) \arrow[r, "\bar{i}^*"] \arrow[d, "\cong", "p_\natural"'] & H^n(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) \arrow[d, "\bar{p}_\natural"] & H^n(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) \arrow[l, "\bar{j}^*"', "\cong"] \arrow[d, "\bar{\pi}_\natural"] \\
H^0(B_H) \arrow[r, "i^*", "\cong"'] & H^0(B_H) & H^0(\bar{B}_H) \arrow[l, "j^*"', "\cong"]
\end{tikzcd}
$$

Note that $p_\natural$ is an isomorphism because the fibre of the fibre pair $(E_G \times_H M, E_G \times_H (M - *)) \xrightarrow{p} B_H$ is $(M, M - *)$ which has the cohomology of $(R^n, R^n - 0)$; thus the spectral sequence for p takes a very simple form, and $p_\natural$ may be thought of as the Thom isomorphism.

Now we define $\underline{U} \in H^n(E_G \times_H M, E_G \times_H (M - *))$ by the equation $p_\natural(\underline{U}) = 1$. Define $\underline{U}_1 \in H^n(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H))$ by $\underline{U}_1 = (\bar{j}^*)^{-1}\bar{i}^*(\underline{U})$. Then $\bar{\pi}_\natural(\underline{U}_1) = 1 \in H^0(\bar{B}_H)$ by diagram (2).

We have the fibre square

$$
(3)\qquad
\begin{tikzcd}
(M, \dot{M}) \arrow[r] \arrow[d] & (M, \dot{M}) \arrow[d] \\
M \times (M, \dot{M}) \arrow[r] \arrow[d] & (\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) \arrow[d] \\
M \arrow[r, "i"] & \bar{B}_H
\end{tikzcd}
$$

arising from the fibre inclusion $M \xrightarrow{i} \bar{B}_H \to B_G$, and restricting diagram (2) to the bundles over the fibres yields

$$
(4)\qquad
\begin{tikzcd}
H^n(\dot{M} \times M, \dot{M} \times M - \Delta) \arrow[r, "1 \times i^*"] \arrow[d, "\cong", "p_\natural"'] & H^n(\dot{M} \times M, \dot{M} \times \dot{M}) \arrow[d, "\bar{p}_\natural"] & H^n(M \times M, M \times \dot{M}) \arrow[l, "\bar{j}^*"'] \arrow[d, "\bar{\pi}_\natural"] \\
H^0(\dot{M}) \arrow[r, "\cong"'] & H^0(M^0) \arrow[r, "\cong"'] & H^0(M)
\end{tikzcd}
$$

where Δ denotes the diagonal.

Define $U \in H^n(\dot{M} \times M, M \times M - \Delta)$ by $p_\natural(U) = 1$, and define $U_1 \in H^n(M \times M, M \times \dot{M})$ as image of U.

Now let $T : X \times Y \to Y \times X$ stand for the twisting map. Noting that

$T : \bar{\pi}^*(\bar{B}_H) \to \bar{\pi}^*(\bar{B}_H)$ arises from the restriction of the twisting map to $\bar{\pi}^*(\bar{B}_H) \subset \bar{B}_H \times \bar{B}_H$, we have a commutative diagram:

$$
(5)\qquad
\begin{tikzcd}
(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H)) \arrow[r, "T"] & (\bar{\pi}^*(\bar{B}_H), T(\bar{\pi}^*(\dot{B}_H))) \\
(M \times M, M \times \dot{M}) \arrow[r, "T"] \arrow[u, "\bar{i}"] & (M \times M, \dot{M} \times M) \arrow[u]
\end{tikzcd}
$$

where $\bar{i}$ comes from the fibre square (3).

Define $\underline{U}_2 \in H^n(\bar{\pi}^*(\bar{B}_H), T(\bar{\pi}^*(\dot{B}_H))$ by $\underline{U}_2 = (-1)^n T^*(\underline{U}_1)$. Similarly define $U_2 \in H^n(M \times M, \dot{M} \times M)$. Then the naturality of integration along the fibre and diagram (5) implies that $\underline{U}$, $\underline{U}_1$ and $\underline{U}_2$ defined in the universal case pull back under inclusion to U, U_1 and U_2 defined in the product case.

Now consider $U_1 \cup U_2 \in H^{2n}((M, \dot{M}) \times (M, \dot{M}))$. We have a relative fibre bundle pair

$$(M, \dot{M}) \to (M \times M, (M \times \dot{M}) \cup (\dot{M} \times M)) \xrightarrow{\pi} (M, M)\ ,$$

and we may define integration along the fibre $\pi_\natural : H^i((M, \dot{M}) \times (M, \dot{M})) \to H^{i-n}(M, \dot{M})$. In this simple situation, $\pi_\natural$ is the same as the slant product with the fundamental class $z \in H_n(M, \dot{M})$ (that is, $\pi_\natural(y) = y/z$). We call $\chi = \pi_\natural(U_1 \cup U_2)$ the Euler class in $H^n(M, \dot{M})$. This definition is easily seen to agree with that of Spanier [5, p. 347]. Thus we have $\chi = \chi(M)\mu \in H^n(M, \dot{M})$ where μ is the appropriately chosen generator.

On the other hand we have

$$\underline{U}_1 \cup \underline{U}_2 \in H^{2n}(\bar{\pi}^*(\dot{B}_H), \bar{\pi}^*(\bar{B}_H) \cup T(\bar{\pi}^*(\dot{B}_H)))\ .$$

Note that $T(\bar{\pi}^*(\dot{B}_H)) = \bar{\pi}^{-1}(\dot{B}_H)$. Thus we are lead to consider the relative fibre bundle pair

$$(M, \dot{M}) \to (\bar{\pi}^*(\dot{B}_H), \bar{\pi}^*(\bar{B}_H) \cup \bar{\pi}^{-1}(\dot{B}_H)) \xrightarrow{\pi} (B_H, \dot{B}_H)\ .$$

Thus we have integration along the fibre

$$\bar{\pi}_\natural : H^i(\bar{\pi}^*(\bar{B}_H), \bar{\pi}^*(\dot{B}_H) \cup \bar{\pi}^{-1}(\dot{B}_H)) \to H^{i-n}(\bar{B}_H, \dot{B}_H)\ .$$

Define the Euler class $\chi = \bar{\pi}_\natural(U_1 \cup U_2) \in H^n(\bar{B}_H, \dot{B}_H)$. By naturality of $\bar{\pi}_\natural$, we see that $i^*(\chi) = \chi(M)\mu$ for $i : (M, \dot{M}) \to (\bar{B}_H, \dot{B}_H)$, the fibre inclusion.

Since $(M, \dot{M}) \to (\bar{B}_H, \dot{B}_H) \xrightarrow{\bar{\pi}} B_G$ is the universal bundle pair for bundle pairs $(M, \dot{M}) \to (E, \dot{E}) \to B$ with structural group preserving the orientation of $(M, \dot{M})$, we always can find a fibre square

$$
(6)\qquad
\begin{tikzcd}
(M, \dot{M}) \arrow[r, "1"] \arrow[d, "i"] & (M, \dot{M}) \arrow[d, "i"] \\
(E, \dot{E}) \arrow[r, "\bar{f}"] \arrow[d, "\pi"] & (\bar{B}_H, \dot{B}_H) \arrow[d, "\pi"] \\
B \arrow[r, "f"] & B_G .
\end{tikzcd}
$$

Define $\chi \in H^n(E, \dot{E})$ by $\chi = \bar{f}^*(\chi)$. It is clear that $i^*(\chi) = \chi(M)\mu$, so Theorem D is proved.

Note that every possible $\bar{f}$ which arises in diagram (6) must be fibrewise homotopic to any other [4], so χ is uniquely defined.

4. Proof of Theorem A

It is clear that Theorem A would be true in general if we can prove Theorem A for the case where G is the identity component of the group of homeomorphisms of M. So we make that assumption.

First we shall prove Theorem A when M is an oriented manifold. We have the fibre square

$$
(7)\qquad
\begin{tikzcd}
G \times (M, \dot{M}) \arrow[r, "\hat{\omega}"] \arrow[d, "i \times 1"] & (M, \dot{M}) \arrow[d] \\
E_G \times (M, \dot{M}) \arrow[r, "\phi"] \arrow[d] & (E_G \times_G M, E_G \times_G \dot{M}) \arrow[d] \\
B_G \arrow[r] & B_G
\end{tikzcd}
$$

where $\hat{\omega}$ is the action of G on M, and ϕ takes $(e, x) \mapsto \langle e, x \rangle$. Since G is connected, we may apply Theorem D to the fibration on the right. Thus $\hat{\omega}^*(\chi(M)\mu)$ $(i \times 1)^*\phi^*(\chi)$. Since E_G is contractible, we see that

$$\hat{\omega}^*(\chi(M)\mu) = 1 \times (\chi(M)\mu) \in H^n(G \times (M, \dot{M}); Z) .$$

Let $\alpha \in H^i(M; G)$ be any element for $i > 0$. Then $\alpha \cup (\chi(M)\mu) \in H^{n+i}(M, \dot{M}; G)$ $\cong 0$. Thus

$$
\begin{aligned}
0 &= \hat{\omega}^*(\alpha \cup (\chi(M)\mu) = \hat{\omega}^*(\alpha) \cup (\hat{\omega}^*(\chi(M)\mu) \\
&= ((\omega^*(\alpha) \times 1) + \text{other terms}) \cup (1 \times (\chi(M)\mu) \\
&= (\omega^*(\alpha) \times (\chi(M)\mu)) + (\text{other terms}) \cup (1 \times \chi(M)\mu) \\
&= \omega^*(\alpha) \times (\chi(M)\mu) = \chi(M)\omega^*(\alpha) \times \mu .
\end{aligned}
$$

Hence $\chi(M)\omega^*(\alpha) = 0$ when M is oriented.

Now we assume that M is unoriented. Let $\tilde{M}$ be the oriented double covering of M, and D the mapping cylinder of the projection $\tilde{M} \to M$. Then D is a manifold with boundary. We may think of G as acting on $\tilde{M}$ by lifting every homeomorphism $h : M \to M$ to that lifting $\tilde{h} : \tilde{M} \to \tilde{M}$ which preserves orientation. Then G acts on D as a group of homeomorphisms by $g(x, t) = (\tilde{g}(t), t)$.

Thus we obtain the following commutative diagram:

$$
(8)\qquad
\begin{tikzcd}
G \arrow[d, "\omega"] \arrow[dr, "\omega"] & \\
M \arrow[r, "i"] & D
\end{tikzcd}
$$

Since the inclusion i is a homotopy equivalence, Theorem A holds for $G \xrightarrow{\omega} M$ if it holds for $G \xrightarrow{\omega} D$. But this is the case as follows from the following lemma.

Lemma 3. *D is orientable, and G preserves the orientation.*

Proof. First assume that M is closed. Then $\dot{D} = \tilde{M}$ and is orientable. An examination of the homology exact sequence of the pair $(D, \dot{D})$ shows that $H_{n+1}(D, \dot{D}) \cong Z$. So D is orientable.

Now assume that M has nonempty boundary $\dot{M}$. Then $\dot{D} = \tilde{M} \cup D(\dot{M})$ where $D(\dot{M})$ is the mapping cylinder of $\tilde{M} \xrightarrow{p} M$ restricted to $\partial\tilde{M} \to \dot{M}$. Now either $D(\dot{M})$ is $\dot{M} \times I$ in case $\dot{M}$ is orientable or it is the mapping cylinder of the bundle covering of $\dot{M}$. In either case $D(\dot{M})$ is orientable. Thus $\dot{D}$ is orientable. Then the homology exact sequence of $(D, \dot{D})$ implies that D is orientable. It is easily seen that G preserves the orientation.

5. Proof of Theorem B

We first proves Theorem B for the case when M is connected and orientable and $\pi_1(B)$ operates trivially on $H^n(M^n, \dot{M}) \cong Z$ in the fibration $(M, \dot{M}) \to (E, \dot{E}) \xrightarrow{\pi} B$.

Define $\tau : H^*(E, p^{-1}(L); G) \to H^*(B; G)$ by letting $\tau(\alpha) = \pi_\natural(\alpha \cup \chi)$.

Lemma 4. *$\tau \circ p^*(\alpha) = \chi(M)\alpha$ for all $\alpha \in H^*(B, L; G)$.*

Proof. From the fibre square

$$
(9)\qquad
\begin{tikzcd}
(M, \dot{M}) \arrow[r, "1"] \arrow[d, "1"] & (M, \dot{M}) \arrow[d] \\
(M, \dot{M}) \arrow[r, "i"] \arrow[d, "\pi'"] & (E, \dot{E}) \arrow[d, "\pi"] \\
* \arrow[r, "c"] & B
\end{tikzcd}
$$

we have $\pi_\natural(\chi) = \pi'_\natural i^*(\chi)$ by identifying $H^0(*)$ with $H^0(B)$. So $\pi_\natural(\chi) = \pi'_\natural(i^*(\chi)) = \pi'_\natural(\chi(M)\mu) = \chi(M)\pi'_\natural(\mu) = \chi(M)1$. Hence $\tau \circ p^*(\alpha) = \pi_\natural(p^*(\alpha) \cup \chi) = \alpha \cup \pi_\natural(\chi) = \alpha \cup (\chi(M)1) = \chi(M)\alpha$.

From now on we shall surpress L and $p^{-1}(L)$ in our notation.

Next we shall show Theorem B is true for M unoriented and connected. Let D be the mapping cylinder as in diagram (8). The projection $q : D \to M$ is equivariant with respect to the action of G. Thus we get a fibre square

$$\begin{tikzcd} D \arrow[r, "q"] \arrow[d] & M \arrow[d] \\ \bar{E} \arrow[r, "\tilde{q}"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\ B \arrow[r, "1"] & B . \end{tikzcd} \tag{10}$$

The left fibration satisfies the previous case since D is oriented and G preserves the orientation by Lemma 3, so there exists a transfer $\tau_1 : H^*(\bar{E}; G) \to H^*(B; G)$. Define $\tau : H^*(E; G) \to H^*(B; G)$ by $\tau = \tau_1\tilde{q}^*$. Then $\tau \circ p^* = \tau_1\tilde{q}^*p^* = \tau_1 p_1^* = \chi(D)1 = \chi(M)1$.

Now we assume that M is orientable and connected but that $\pi_1(B)$ does not act trivially on $H^n(M, \dot{M}; Z)$. Then we obtain the commutative diagram

$$\begin{tikzcd} M \times P^2 \arrow[r, "\pi"] \arrow[d] & M \arrow[d] \\ E \times P^2 \arrow[r, "\pi"] \arrow[d, "p_1"] & E \arrow[d, "p"] \\ B \arrow[r, "1"'] & B \end{tikzcd}$$

where P^2 is the real projective plane, and π is projection on the first factor. The fibre bundle on the left satisfies the above case since $M \times P^2$ is unorientable. Thus there exists a transfer $\tau_1 : H^*(E \times P^2; G) \to H^*(B; G)$. Define $\tau : H^*(E; G) \to H^*(B; G)$ by $\tau = \tau_1\pi^*$. Then $\tau \circ p^* = \tau_1\pi^*p^* = \tau_1 p_1^* = \chi(M \times P^2)1 = \chi(M)1$.

Now assume that M is not connected. Then the fibre bundle $M \to E \xrightarrow{p} B$ factors through the fibre bundles $E \xrightarrow{p_2} \tilde{B} \xrightarrow{p_1} B$, where $\tilde{B}$ is an N-fold covering of B, and M is N disjoint copies of M_0. Thus we have a transfer for $M_0 \to E \xrightarrow{p_2} \tilde{B}$; call it τ_2. Also we have the classical transfer for the covering τ_1. Define $\tau : H^*(E; G) \to H^*(B; G)$ by $\tau = \tau_1 \circ \tau_2$. Then $\tau \circ p^* = \tau_1 \circ \tau_2 \circ p_2^* \circ p_1^*$

$= \tau_1 \circ \chi(M_0)1 \circ p_1^* = \chi(M_0)\tau_1 \circ p_1^* = N\chi(M_0)1 = \chi(M)1$.

In the case where E is not connected, we obtain a transfer for each component of E. Then we sum them to obtain the transfer for $E \xrightarrow{p} B$. Finally, if B is not connected, (we assume that each fibre of $E \xrightarrow{p} B$ is M), then the direct sum of the transfers over each component of B will yield the transfer we seek.

6. Proof of Theorem C and remarks

We begin as before, by assuming that E and M are connected and M is orientable, and that $\pi_1(B)$ preserves orientation. Then we have the Euler class $\chi \in H^n(E, \dot{E})$. Define the transfer $\tau : H_*(B, L; G) \to H_*(E, \pi^{-1}(L); G)$ by $\tau(\alpha) = \chi \cap \pi^\natural(\alpha)$ where $\pi^\natural : H_*(B, L; G) \to H_*(E, \dot{E} \cup \pi^{-1}(L); G)$. Then $p_* \circ \tau(\alpha) = p_*(\chi \cap \pi^\natural(\alpha)) = \pi_\natural(\chi) \cap \alpha = \chi(M)1 \cap \alpha = \chi(M)\alpha$.
The remainder of the proof is dual to § 5.

Several remarks are in order.

1. Various other transfers may be defined based on characteristic numbers of a manifold, however, not in the generality as the one we have defined. The essential point is to find the appropriate version of Theorem D. For example, if M^n is a closed connected differential manifold, $M \xrightarrow{i} E \xrightarrow{p} B$ is a fibre bundle with structural group G acting differentially on M, and M has a non-zero Pontryagin number p_I, then there is a class $\nu \in H^n(E; Z)$ such that $i^*(\nu) = p_I\mu$. Then we may prove, as before, that $p_I\omega^* = 0$ where $\omega : G \to M$ is the evaluation map from the structural group G, and obtain transfer theorems but only under the above restricted hypothesis. To see that $p_I\mu$ is in the image of i^*, we follow the idea of Borel [2, Lemma 3.2]. Similarly, for M a closed connected topological manifold we may define transfers (in Z_2 coefficients) by using Stiefel-Whitney numbers.
2. Theorem D is true for Z_2 coefficients with no orientability condition on M or the fibre bundle.
3. The Euler-Poincaré number in Theorems A, B and C is essential. For example, $O(3)$ acts on S^2 and it is well known that $\omega^* : \tilde{H}^*(S^2; Z_2) \to \tilde{H}^*(O(3); Z_2)$ is not that trivial homomorphism. But $\chi(S^2)\omega^* = 2\omega^* = 0$ since 2 is zero in Z_2. An example in the case of the transfer comes from the universal principal bundle $G \xrightarrow{i} E_G \xrightarrow{p} B_G$. Here $\tau \circ p^* = \chi(G)1$. But $\tilde{H}^*(E_G) = 0$. So $\chi(G) = 0$.
4. Applications will appear elsewhere. Among them they include the fact that RP^{2n} or CP^{2n} or QP^{2n} or *Cayley* p^2 do not fibre with a manifold as a fibre.

References

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