The fact that the horizontal rows are fibrations will imply that $1 \times p\colon E_G \times E \to E_G \times X$ is fibre homotopy equivalent to the pullback of "$\overline{p}$ made into a fibration." Since E_G is contractible $E \overset{p}{\to} X$ is a pullback of $1 \times p$, hence p is fibre homotopy equivalent to the pullback of "$\overline{p}$ made into a fibration" as required.

Now we prove a special case of theorem 1.

Lemma 3: Suppose G acts freely on X. Then G lifts up to homotopy if and only if $E \overset{p}{\to} X$ is fibre homotopy equivalent to a pullback by i of a fibration over X_G.

Proof: Lemma 2 implies half of the theorem. Now assume that $E \overset{p}{\to} X$ is a pullback of a fibration over X_G. It is well known that X_G is homotopy equivalent to the quotient space X/G when the action of G is free. This can be seen as follows: The projection $E_G \times X \to X$ is equivariant with respect to the diagonal action of G on $E_G \times X$, hence passing to quotients, we have bundle $E_G \to E_G \times_G X \overset{\psi}{\to} X/G$; and since E_G, the fibre, is contractible the map $X_G = E_G \times_G X \to X/G$ must be a homotopy equivalence. Now the quotient map ρ factors as follows: $\rho\colon X \overset{i}{\to} X_G \overset{\psi}{\to} X/G$.

Now $E \overset{p}{\to} X$ must be fibre homotopy equivalent to a pullback by ρ of a fibration over X/G since ψ is a homotopy equivalence. So suppose, without loss of generality