Now $SO(k)$ acts effectively on CP^n, hence $Spin(k)$ acts on CP^n via the double covering $Spin(k) \to SO(k)$. Now this action of $Spin(k)$ must lift to some action of $Spin(k)$ on S^{2n-1} since $Spin(k)$ is simply connected. Now this lifted action on S^{2n-1} is either effective, or else the subgroup Z_2 which is the kernel of $p: Spin(k) \to SO(k)$ acts trivially. In the latter case, since $SO(k)$ acts effectively, the $Spin(k)$ action on S^{2n-1} gives rise to an $SO(k)$ action on S^{2n-1} lifting the original action on CP^n. Thus when the inequalities of the hypotheses hold, there can be no effective action of $Spin(k)$ on S^{2n-1} and so the latter case must always hold and thus there is a lifting of $SO(k)$.

§ 5. Equivariance Up to Homotopy

Given a map $f: Y \to X$ and an action of G on X, we ask whether it is possible to impose a G action on some Y', which is homotopy equivalent to Y, such that there is an equivariant map $f': Y' \to X$ such that the diagram

$$\begin{array}{ccc} Y & \xrightarrow{f} & X \\ {\scriptstyle h}\downarrow & \nearrow_{f'} & \\ Y' & & \end{array}$$

commutes up to homotopy where h is some homotopy equivalence.