

Proof: If $i \sim \text{constant}$, then any pullback of a fibration over X_G by i must be fibre homotopic to a trivial fibration $F \times X \rightarrow X$. Now suppose that the only equivariant maps to X are homotopy equivalent to projections $F \times X \rightarrow X$. Then the pullback by i of any fibration over X_G must be a trivial fibration. For, consider the path space fibration over X_G , namely $\Omega X_G \rightarrow P \rightarrow X_G$. This pulls back to a principal fibration of the form $\Omega X_G \rightarrow \Omega X_G \times X \xrightarrow{\text{Proj.}} X$. But this implies that i is homotopic to a constant since $\Omega X_G \rightarrow P \rightarrow X_G$ is universal for principal X_G fibrations. This proves the first part of the proposition. Now X must be an H-space since any fibre which is homotopic to a constant in the total space of a fibration must be an H-space.

An example of such an action is provided by G acting on G by left multiplication. Then we have $G \rightarrow (G)_G = E_G \rightarrow B_G$ and so $i \sim \text{constant}$.

Now we ask the converse question: Are there actions for which every map is equivariant up to homotopy? The trivial action of G on X is one such. We will find a more interesting example.

We recall that E_G is a contractible group. Then we define a cononical action of G on X_G by letting $g(\langle e, x \rangle_G) = \langle g \cdot e, x \rangle_G$. Here we regard G as a closed subgroup of E_G and " \cdot " denotes group multiplication. Here